EXT AND KOSZUL ALGEBRAS

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This paper is based almost entirely on my talks at the Morelia workshop in the Summer of 1998. Most of the results presented here are basic and can be found in the literature, for instance in the papers of Beilinson, Ginsburg and Soergel ([BGS]), or Green and Martínez-Villa ([GM1] and [GM2]), or S.P.Smith([S]), or Agoston, Dlab and Lukacs ([ADL2]); for that reason many are presented without proofs. Some new results obtained with Green and Martínez-Villa are given in this paper however, and I take this chance to thank my co-authors for allowing me to insert them here. No attempt has been made to discuss the connection with noncommutative geometry; a good starting point would be [S]; similarly, I did not discuss several new results for selfinjective Koszul algebras obtained by Martínez-Villa and Guo.

I am very grateful to Ed Green, Eduardo Marcos and Roberto Martínez-Villa for many interesting and challenging conversations.

1. Notations, definitions and some basic results.

Let K be a field, and let Λ be a K-algebra. We will denote by mod Λ, the category of finitely generated left Λ-modules. For most of our examples, we will use the setup of quivers with relations to describe our algebras, and representations of quivers with relations to describe our modules. The reader is referred to [ARS] for the basic definitions and notations.

Definition 1.1. If M and N are two Λ-modules, an *n-fold extension* of M by N is an exact sequence of Λ-modules

$$0 \to N \to M_{n-1} \to \cdots \to M_0 \to M \to 0$$

(Thus, a short exact sequence $0 \to N \to E \to M \to 0$ is a 1-fold extension.)

For two n-fold extensions of M by N:

$$\xi : 0 \to N \to M_{n-1} \to \cdots \to M_0 \to M \to 0$$

and

$$\eta : 0 \to N \to M'_{n-1} \to \cdots \to M'_0 \to M \to 0$$

we say that "ξ *is related to* η," and write $\xi \rightsquigarrow \eta$ if there is a commutative diagram

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & N & \longrightarrow & M_{n-1} & \longrightarrow & \cdots & \longrightarrow & M_1 & \longrightarrow & M_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle f_{n-1}} & & & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \| & & \\
0 & \longrightarrow & N & \longrightarrow & M'_{n-1} & \longrightarrow & \cdots & \longrightarrow & M'_1 & \longrightarrow & M'_0 & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

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It is easy to see that normally $\rightsquigarrow$ is not an equivalence relation. However, we can look at the equivalence relation generated by $\rightsquigarrow$: we say that ξ is *equivalent* to η, if there is a sequence of n-fold extensions of M by N:

$$\xi = \xi_0, \xi_1, \dots, \xi_k = \eta$$

such that, for each $i = 0, \dots, k-1$ we have $\xi_i \rightsquigarrow \xi_{i+1}$ or $\xi_{i+1} \rightsquigarrow \xi_i$.

Definition 1.2. The set of all the equivalence classes of n-fold extensions of M by N is denoted $\operatorname{Ext}^n_\Lambda(M, N)$.

We can define a K-vector space structure on $\operatorname{Ext}^n_\Lambda(M, N)$, see for instance [Be]. A faster and somehow less technical way of seeing this is by constructing $\operatorname{Ext}^n_\Lambda(M, N)$ using projective (or injective) resolutions. We first take a projective resolution of M:

$$\cdots \to \mathcal{P}_{n+1} \xrightarrow{d_{n+1}} \mathcal{P}_n \xrightarrow{d_n} \cdots \to \mathcal{P}_1 \xrightarrow{d_1} \mathcal{P}_0 \xrightarrow{d_0} M \to 0.$$

Then we construct the complex:

$$0 \to \operatorname{Hom}_\Lambda(\mathcal{P}_0, N) \xrightarrow{d_1^*} \operatorname{Hom}_\Lambda(\mathcal{P}_1, N) \to \dots \xrightarrow{d_{n+1}^*} \operatorname{Hom}_\Lambda(\mathcal{P}_{n+1}, N) \to \dots$$

where $d_i^* = \operatorname{Hom}(d_i, N)$ are the obvious induced morphisms. Then we have

$$\operatorname{Ext}^n_\Lambda(M, N) \cong \frac{\ker d_{n+1}^*}{\operatorname{Im} d_n^*} \tag{1.3}$$

In some special cases we can refine the above construction:

Proposition 1.4. *Let Λ be a finite dimensional K-algebra and let N be a semisimple Λ-module. Let*

$$\dots \mathcal{P}_{n+1} \xrightarrow{d_{n+1}} \mathcal{P}_n \to \dots \quad \to \mathcal{P}_0 \to M \to 0$$

be a minimal projective resolution of M. Then, for each $n \geq 0$ $\operatorname{Ext}^n_\Lambda(M, N) \cong \operatorname{Hom}_\Lambda(\mathcal{P}_n, N)$.

Proof. The minimality of the resolution implies that for each n, $\operatorname{Im} d_{n+1} \subseteq \mathbf{r}\mathcal{P}_n$ where $\mathbf{r}$ is the Jacobson radical of Λ.

$$\begin{array}{ccccc} \mathcal{P}_{n+1} & \xrightarrow{d_{n+1}} & \mathcal{P}_n & \xrightarrow{d_n} & \mathcal{P}_{n-1} \\ & & \downarrow{\scriptstyle \xi} & \swarrow{\scriptstyle \eta} & \\ & & N & & \end{array}$$

If $\xi \in \operatorname{Hom}_\Lambda(\mathcal{P}_n, N)$, since N is semisimple, $\ker \xi$ includes $\mathbf{r}\mathcal{P}_n$, so $\xi d_{n+1} = 0$, therefore $\operatorname{Hom}_\Lambda(\mathcal{P}_n, N) = \ker d_{n+1}^*$. Also, each composition $\eta d_n = 0$ for each $\eta \in \operatorname{Hom}_\Lambda(\mathcal{P}_{n-1}, N)$ since $\operatorname{Im} d_n \subseteq \mathbf{r}\mathcal{P}_{n-1}$ so $\operatorname{Im} d_n^* = 0$ for each n. □

We shall use the above identification throughout this paper. Note that 1.4 holds in the more general case when Λ is *semiprimary* (i.e. $\Lambda/\mathbf{r}$ is semisimple and $\mathbf{r}$ is nilpotent). Proposition 1.4 can be also reformulated in the context of graded resolutions over graded algebras as we will see later in this section.

1.5. ***The Yoneda Product.*** We want to be able to multiply extensions, so we want a bilinear, associative map:

$$\operatorname{Ext}^m_\Lambda(L, K) \otimes \operatorname{Ext}^n_\Lambda(M, L) \to \operatorname{Ext}^{m+n}_\Lambda(M, K)$$

There are two equivalent ways of defining such a mapping:

(a) *Splicing:*

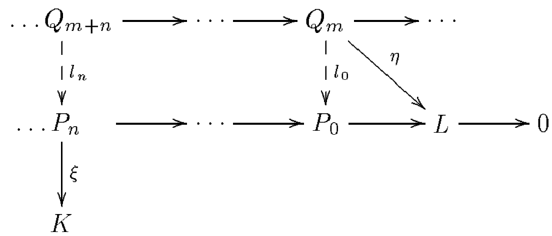
$$\begin{array}{ll} \text{if} \quad \xi = 0 \to K \longrightarrow A_{m-1} \to \cdots \to A_0 \xrightarrow{f} L \to 0 & \text{and} \\ \quad\ \ \eta = 0 \to L \xrightarrow{g} B_{n-1} \to \cdots \to B_0 \longrightarrow M \to 0 & \end{array}$$

are two representatives of elements in $\mathrm{Ext}^m_\Lambda(L,K)$ and $\mathrm{Ext}^n_\Lambda(M,L)$, let

$$\xi\eta : 0 \to K \to A_{m-1} \to \cdots \to A_0 \xrightarrow{gf} B_{n-1} \to \cdots \to B_0 \to M \to 0\cdot$$

Then $[\xi]\cdot[\eta] = [\xi\eta]$, the equivalence class of the $n+m$-fold extension.

(b) *Using projective resolutions.* Let $(\mathcal{P})$ and $(\mathcal{Q})$ be projective resolutions of L and M respectively. Let η, ξ be representatives. We have the following commutative diagram:



where $l_0, l_1, \dots l_n$ denote successive liftings of η.

Then, for $[\xi] \in \mathrm{Ext}^n_\Lambda(L,K)$ and $[\eta] \in \mathrm{Ext}^m_\Lambda(M,L)$ we have

$$[\xi]\cdot[\eta] = [\xi \circ l_n].$$

It is clear that this way of multiplying extensions does not depend on the chosen resolutions, or on the chosen liftings. In fact, if both $(\mathcal{P})$ and $(\mathcal{Q})$ are minimal resolutions, and, if K is a semisimple Λ-module, then $\xi \circ l_n$ is an element of $\mathrm{Ext}^{m+n}_\Lambda(M,K)$.

Example 1.6. Let $\Lambda = KQ$ and $\Gamma = KQ/I$ where Q is the quiver $\overset{1}{\bullet} \xrightarrow{x} \overset{2}{\bullet} \xrightarrow{y} \overset{3}{\bullet}$, and $I = <xy>$ be the ideal generated by the composition of the two arrows. Let $\xi : 0 \to S_3 \to P_2 \to S_2 \to 0$ and let $\eta : 0 \to S_2 \to I_2 \to S_1 \to 0$, where S_i denote the simple modules corresponding to i $(i = 1,2,3)$, and where P_2 and I_2 denote the projective cover (injective envelope respectively) of S_2. It is clear that we may consider such exact sequence over both Λ and Γ, so ξ and η represent elements in $\mathrm{Ext}^1_\Lambda(S_1,S_2)$, $\mathrm{Ext}^1_\Lambda(S_2,S_3)$, as well as in $\mathrm{Ext}^1_\Gamma(S_1,S_2)$ and $\mathrm{Ext}^1_\Gamma(S_2,S_3)$.

If we consider the spliced sequence

$$\xi\eta : 0 \to S_3 \to P_2 \to I_2 \to S_1 \to 0$$

then, $[\xi\eta] = 0$ in $\mathrm{Ext}^2_\Lambda(S_1,S_3)$, but $[\xi\eta] \neq 0$ in $\mathrm{Ext}^2_\Gamma(S_1,S_3)$. It is probably easier to notice these facts by using the definition of $[\xi]\cdot[\eta]$ using projective resolutions, rather than computing the equivalence class of the 2-fold extension $\xi\eta$ over the algebra Γ.

1.7. In this article, by a graded K-algebra we mean a K-algebra $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \dots$, where for each $i \geq 0$, Λ_i is a finite dimensional K-vector space, $\Lambda_0 = K \times \cdots \times K$ is the product of finitely many copies of K and, for each i and j we have $\Lambda_i\Lambda_j \subseteq \Lambda_{i+j}$. If Λ is a graded algebra, M is a graded Λ-module if $M = \bigoplus_{i \geq i_0} M_i$, where for each k and j we have $\Lambda_k M_j \subseteq M_{k+j}$, and each component M_i is a K-vector space. A graded Λ module $M = \oplus_{i \geq k} M_i$ is *generated in degree* k if for each $j \geq k$ we have $M_j = \Lambda_{j-k} M_k$. If $\Lambda = \oplus_{i \geq 0} \Lambda_i$ is generated as a Λ-module in degree 0, then $\Lambda_i \Lambda_j = \Lambda_{i+j}$ for all i,j and we also say that as an algebra, Λ is generated in degrees 0 and 1. A degree 0 morphism between two graded modules M and N is a Λ-homomorphism f, satisfying $f(M_i) \subseteq N_i$ for each integer i. If Λ is a graded algebra, we denote by $gr\Lambda$ the category of finitely generated graded Λ-modules and degree 0 maps.

By a graded projective resolution of the graded module M we mean a resolution

$$\dots \mathcal{P}_n \xrightarrow{d_n} \mathcal{P}_{n-1} \to \cdots \to \mathcal{P}_1 \xrightarrow{d_1} \mathcal{P}_0 \xrightarrow{d_0} M \to 0$$

where, for each $i \geq 0$, $\mathcal{P}_i$ is a graded projective Λ-module and each map d_i is a degree 0 morphism. It is well known that a graded projective resolution of a module is also a projective resolution in the usual sense. If $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \dots$ is a graded K-algebra, we denote by $\mathbf{r} = \Lambda_1 \oplus \Lambda_2 \oplus \dots$ its *graded radical*. A graded projective resolution

$$\dots \mathcal{P}_n \xrightarrow{d_n} \mathcal{P}_{n-1} \xrightarrow{d_{n-1}} \dots \to \mathcal{P}_1 \xrightarrow{d_1} \mathcal{P}_0 \xrightarrow{d_0} M \to 0$$

is *minimal* if, for each $n \geq 1$ we have $\operatorname{Im} d_n \subseteq \mathbf{r}\mathcal{P}_{n-1}$ is a homogeneous submodule.

It is easy to see that if Λ is a graded K-algebra and M and N are finitely generated graded Λ-modules, we have $\operatorname{Hom}_\Lambda(M, N) = \bigoplus_{d \in \mathbb{Z}} \operatorname{Hom}_{gr\Lambda}(M, N(d))$ where $N(d)$ denotes the d-shift of N, that is, for each j, $N(d)_j = N_{d+j}$. If in addition the module N is semisimple and if

$$\dots \to \mathcal{P}_n \to \mathcal{P}_{n-1} \to \dots \to \mathcal{P}_0 \to M \to 0$$

is a minimal projective graded resolution of M, then we can still use 1.4 to compute $\operatorname{Ext}^n_\Lambda(M, N)$, namely we obtain that for each $n \geq 0$, $\operatorname{Ext}^n_\Lambda(M, N) \cong \bigoplus_{d \in \mathbb{Z}} \operatorname{Hom}_{gr\Lambda}(\mathcal{P}_n, N(d))$.

1.8. **The Ext-algebra of Λ.** Let Λ be a K-algebra and let $\mathbf{r}$ be its radical (Jacobson radical if Λ is semiprimary, or the graded radical is the graded case). The *Ext-algebra* $\mathrm{E}(\Lambda)$, also called the *Yoneda algebra* of Λ, is the algebra:

$$\mathrm{E}(\Lambda) = \bigoplus_{n \geq 0} \operatorname{Ext}^n_\Lambda(\Lambda/\mathbf{r}, \Lambda/\mathbf{r})$$

where the multiplicative structure is given by the Yoneda product

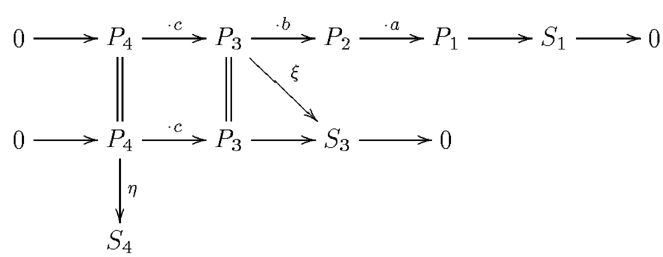
$$\operatorname{Ext}^m_\Lambda(\Lambda/\mathbf{r}, \Lambda/\mathbf{r}) \otimes \operatorname{Ext}^n_\Lambda(\Lambda/\mathbf{r}, \Lambda/\mathbf{r}) \to \operatorname{Ext}^{m+n}_\Lambda(\Lambda/\mathbf{r}, \Lambda/\mathbf{r})$$

Remarks. (i) Assume that the algebra Λ if finite dimensional. Then $\mathrm{E}(\Lambda)$ need not be finite dimensional, in fact $\mathrm{E}(\Lambda)$ need not even be noetherian. It is however easy to prove in this case that $\dim_K \mathrm{E}(\Lambda) < \infty$ if and only if $\operatorname{gldim} \Lambda < \infty$.

(ii) The Ext-algebra $\mathrm{E}(\Lambda)$ is usually noncommutative.

Example 1.9. Let $\Lambda = KQ/I$ where Q is the quiver $x \circlearrowleft \bullet \circlearrowright y$ and $I = < x^2, xy, yx, y^2 >$. Then $\mathrm{E}(\Lambda) = K < x, y >$ is the free algebra in the variables x and y.

Example 1.10. Let $\Lambda = KQ/I$ where Q is the quiver $\overset{1}{\bullet} \xrightarrow{a} \overset{2}{\bullet} \xrightarrow{b} \overset{3}{\bullet} \xrightarrow{c} \overset{4}{\bullet}$ and let $I = < ba, cb >$. Let S_i denote the simple representations associated to the vertices i $(i = 1, 2, 3, 4)$, and let P_i denote their projective covers. Let $[\xi] \in \operatorname{Ext}^2_\Lambda(S_1, S_3)$ and let $[\eta] \in \operatorname{Ext}^1_\Lambda(S_3, S_4)$. We compute $[\xi] \cdot [\eta]$ and $[\eta] \cdot [\xi]$ using projective resolutions. We have the following commutative diagram



where the top row is a minimal projective resolution of S_1, the bottom row is a minimal projective resolution of S_3, and the indicated maps are multiplications by the given arrows. Here we may take identity maps for liftings therefore $[\eta][\xi] \neq 0$ in $\operatorname{Ext}^3_\Lambda(S_1, S_4)$. To compute $[\xi] \cdot [\eta]$ we need not use any commutative diagrams since it is clear from the definition in this case that $[\xi] \cdot [\eta] = 0$.

Definition 1.11. For a Λ-module M, we let $E(M) = \bigoplus_{n\geq 0} \operatorname{Ext}^n_\Lambda(M, \Lambda/\mathbf{r})$. It is clear that $E(M)$ is a graded left $\mathrm{E}(\Lambda)$-module via the map

$$\operatorname{Ext}^m_\Lambda(\Lambda/\mathbf{r}, \Lambda/\mathbf{r}) \otimes \operatorname{Ext}^n_\Lambda(M, \Lambda/\mathbf{r}) \to \operatorname{Ext}^{m+n}_\Lambda(M, \Lambda/\mathbf{r}).$$

In this way we obtain an additive contravariant functor

$$E\colon \quad \operatorname{mod}\Lambda \to Gr\mathrm{E}(\Lambda)$$

where $Gr\mathrm{E}(\Lambda)$ denotes the category of graded left $\mathrm{E}(\Lambda)$-modules and degree 0 maps.

2. Examples and computations of Ext-algebras

In this section we describe the Ext-algebra for two classes of algebras: monomial algebras and biserial algebras of finite representation type. For monomial algebras we use the definition of the Yoneda product using minimal projective resolutions, while for biserial algebras we use splicings of multiple fold extensions. We freely quote from [GHZ], [GZ], [GKK] in describing the Ext-algebra of a monomial algebra, and from Brown's preprint ([Br]) for the representation finite biserial case.

2.1. **Monomial algebras.** Let $\Lambda = KQ/I$ where $I =< \rho_1, \ldots, \rho_k >$ and assume further that $\rho_1, \ldots, \rho_k$ are paths in Q and that $\{\rho_1, \ldots, \rho_k\}$ is a minimal set of generators of I in the sense that for each i, ρ_i is not contained in any other ρ_j. We define inductively sets of paths in Q as follows: $Q_0 =$ the vertices of Q, $Q_1 =$ the arrows of Q, $Q_2 = \{\rho_1, \ldots, \rho_k\}$. Assume that we have constructed $Q_0, Q_1, \ldots, Q_i$. A path p in Q is called an *i-prechain* if $p = srq$ where $s \in Q_{i-1}$, $sr \in Q_i$, s has nontrivial image in Λ and rq has zero image in Λ.

$$p : \bullet \underset{q}{\rightsquigarrow} \bullet \underset{r}{\rightsquigarrow} \bullet \underset{s}{\rightsquigarrow} \bullet$$

Let Q_{i+1} be the set of all the i-prechains with the property that no initial subpath is an i-prechain. Clearly $Q_i \cap Q_j = \emptyset$ whenever $i \neq j$. For each $i \geq 0$, let $\mathcal{P}_i = \coprod_{p \in Q_i} \Lambda e_p$ where e_p is the primitive idempotent corresponding to the terminus of p.

Theorem 2.2. *Let Λ be a monomial algebra and let $\mathcal{P}_i$ be defined as above. Then, there exists a minimal projective resolution of $\Lambda/\mathbf{r}$ of the form:*

$$\cdots \to \mathcal{P}_n \to \cdots \to \mathcal{P}_0 \to \Lambda/\mathbf{r} \to 0.$$

Recall that $\operatorname{Ext}^i_\Lambda(\Lambda/\mathbf{r}, \Lambda/\mathbf{r}) = \operatorname{Hom}_\Lambda(\mathcal{P}_i, \Lambda/\mathbf{r})$. This gives an identification of Q_i with a K-basis of $\operatorname{Ext}^i_\Lambda(\Lambda/\mathbf{r}, \Lambda/\mathbf{r})$ via the map $p \mapsto f_p$ where $f_p : \mathcal{P}_i \to \Lambda/\mathbf{r}$ is the morphism taking Λe_q to zero if $p \neq q$ in Q_i, and $f_p(\lambda e_p) = \bar{\lambda}\bar{e}_p$ where $\bar{\lambda}$ denotes the image in $\Lambda/\mathbf{r}$. Using this identification we have the following:

Theorem 2.3. *The family $\{f_p\}_{p \in \bigcup_{i\geq 0} Q_i}$ is a multiplicative K-basis of $\mathrm{E}(\Lambda)$ in the sense that, if $p \in Q_i$ and $q \in Q_j$*

$$f_q \cdot f_p = \begin{cases} f_{qp} & \text{if } qp \in Q_{i+j} \\ 0 & \text{otherwise.} \end{cases}$$

Note that 2.3 gives a criterion for the finite generation of $\mathrm{E}(\Lambda)$ in the monomial algebra case:

Theorem 2.4. $\mathrm{E}(\Lambda)$ *is finitely generated if and only if there exists a positive integer N such that for each $n \geq N$ we may write every path $p \in Q_n$ as $p = q_1 \ldots q_t$, $q_j \in Q_{i_j}$ and $i_1 + \cdots + i_t = n$.*

It is easy now to generate examples of monomial algebras whose Ext algebras are not finitely generated:

Example 2.5. Let $\Lambda = KQ/I$ where Q is the quiver  and let $I =< x^2, y^2, xyx >$. Then $\mathrm{E}(\Lambda)$ is not finitely generated since no path of the form $x(yxyx \dots xy)x$ can be written as a composition of paths in various $Q_i - s$.

2.6. **Biserial algebras.** Recall that a finite dimensional algebra Λ is *biserial* if the radical of indecomposable left or right projective Λ-module is a sum of at most two uniserial submodules whose intersection is either zero or a simple module. The following theorem characterizes biserial algebras of finite representation type in terms of the structure of their almost split sequences:

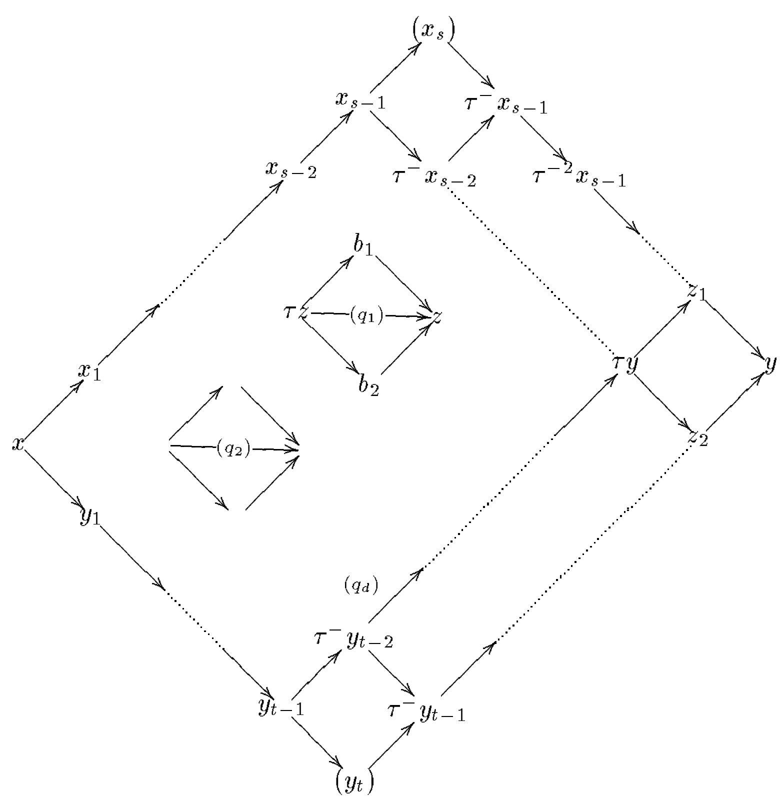
Theorem 2.7. ([AR], [SW]) *Let Λ be a finite dimensional algebra of finite representation type over an algebraically closed field. Then Λ is biserial if and only if $\beta(\Lambda) \leq 2$ where $\beta(\Lambda)$ denotes the largest number of nonprojective, noninjective indecomposable summands appearing in the middle term of an almost split sequence.*

Brown's method ([Br]) of computing a multiplicative basis of $\mathrm{E}(\Lambda)$ in the representation finite biserial case reduces to computing bases for multiple fold extensions, using the geometry of the Auslander-Reiten quiver. We present here the simply connected case. Let Γ_Λ be the Auslander-Reiten quiver of Λ.

Definition 2.8. If x, y are two vertices of Γ_Λ, let $\mathcal{R}(x, y)$ denote the full translation subquiver of Γ_Λ whose vertices lie on a path from x to y. By definition x, y are both in $\mathcal{R}(x, y)$. Then $\mathcal{R}(x, y)$ is a *rectangle* if the subquiver $\mathcal{R}(x, \tau y)$ is nonempty and satisfies the following properties, where τ denotes the Auslander-Reiten translate:

$\mathcal{R}(x, \tau y)$ contains no triangular mesh
$\mathcal{R}(x, \tau y)$ contains no injective vertex except possibly nonuniserial projective-injective vertices.

In general, a rectangle $\mathcal{R}(x, y)$ is of the form



The vertices x_s and y_t are in parentheses to indicate the possibility of a triangular mesh starting at x_{s-1} or y_{t-1}. Given a rectangle $\mathcal{R}(x,y)$, there is a nonsplit exact sequence $0 \to x \to z \to y \to 0$ for each noninjective vertex x. The construction is quite explicit, and we denote this sequence by $\eta(x,y)$.

We now present a way of constructing a basis of the Ext-spaces. It turns out that splicing certain multiple-fold extensions works out quite well in our case. Namely, let $\mathcal{R}(x_m,\dots,x_0)$ be the full subquiver of Γ_Λ whose vertices lie on a path from x_m to x_0 and passing consecutively through the vertices $x_m, x_{m-1},\dots,x_1$ and x_0. We call $\mathcal{R}(x_m,\dots,x_0)$ a *rectangular m-chain* if each of the subquivers $\mathcal{R}(x_i,x_{i-1})$ is a rectangle, and we say that $\mathcal{R}(x_m,\dots,x_0)$ is a *left-maximal rectangular m-chain*, if, in addition, no rectangle $\mathcal{R}(x_i,x_{i-1})$ is a proper subquiver of a rectangle $\mathcal{R}(x,x_{i-1})$. It turns out that if $\mathcal{R}(x_m,\dots,x_1)$ is a left-maximal rectangular m-chain, then by splicing together the short exact sequences $\eta(x_m,x_{m-1}),\eta(x_{m-1},x_{m-2}),\dots,\eta(x_1,x_0)$ we obtain a m-fold extension $\eta(x_m,x_{m-1},\dots,x_0)$ that represents a nonzero element of $\mathrm{Ext}^m(x_0,x_m)$.

Theorem 2.9. *Let $\mathcal{R}^m_{x,y}$ be the collection of all exact sequences $\eta(x_m,\dots,x_0)$ such that $\mathcal{R}(x_{m-1},\dots,x_0)$ is left maximal, where $x_m = x$ and $x_0 = y$ with y simple. Then $\mathcal{R}^m_{x,y}$ is a k-basis for $\mathrm{Ext}^m(y,x)$.*

It turns out that the union $\bigcup_{m\geq 0}\mathcal{R}^m_{x,y}$ when x,y run over the set of simple vertices is a multiplicative k-basis of the Ext-algebra $\mathrm{E}(\Lambda)$. In our notation $\mathcal{R}^0_{x,y}$ denotes a k-basis of $\mathrm{Hom}(x,y)$. Theorem 2.9 can be extended to the general representation finite biserial case using covering techniques. For another approach in describing extensions spaces using the geometry of the Auslander-Reiten quiver see [BC].

3. Koszul algebras

3.1. **The quiver of** $\mathrm{E}(\Lambda)$**.** We start this section by describing the quiver of the Ext-algebra $\mathrm{E}(\Lambda)$. If Λ is finite dimensional, let $S_1,S_2,\dots,S_n$ be a complete set of nonisomorphic simple Λ-modules and let $P_1,P_2,\dots,P_n$ denote their projective covers. For the graded case we assume that $S_1,S_2,\dots,S_n$ are also graded and generated in degree zero and also that $P_1,\dots,P_n$ are their graded projective covers. We have the following remark: (1) For each $i=1,2,\dots,n$, the $\mathrm{E}(\Lambda)$ graded modules $E(S_i)=\oplus_{n\geq 0}\mathrm{Ext}^n_\Lambda(S_i,\Lambda/\mathbf{r})$ form a complete set of nonisomorphic graded projective modules generated in degree 0. (2) For each $i=1,2,\dots,n$ the modules $E(P_i)=\oplus_{n\geq 0}\mathrm{Ext}^n_\Lambda(P_i,\Lambda/\mathbf{r})$ form a complete set on nonisomorphic graded simple $\mathrm{E}(\Lambda)$-modules generated in degree 0, and the induced maps $E(S_i)\to E(P_i)$ are graded projective covers. (3) The quiver of $\mathrm{E}(\Lambda)$ contains the quiver of Λ as a full subquiver.

Proof. We prove only the third statement. Recall that if Λ is a finite dimensional algebra, in order to find its (left) quiver, we proceed as follows: to each of the simple modules S_i we associate a vertex i, and, $\dim_K \mathrm{Ext}^1_\Lambda(S_i,S_j)=\dim_K \mathrm{Hom}_\Lambda(P_j,\mathbf{r}P_i/\mathbf{r}^2P_i)$ is the number of arrows from i to j. For graded algebras such as $\mathrm{E}(\Lambda)$ we must look at $gr\mathrm{E}(\Lambda)$, more specifically at $\oplus_{d\in\mathbf{Z}}\mathrm{Hom}_{gr\mathrm{E}(\Lambda)}(E(S_j),\mathbf{r}E(S_i)(d)/\mathbf{r}^2E(S_i)(d)) = \oplus_{d\in\mathbf{Z}}\mathrm{Ext}^1_{gr\Lambda}(E(P_i),E(P_j)(d))$ where $E(S_i)(d)$ denotes the shift of $E(S_i)$ as defined in the first section. The dimension of these spaces equals the number of arrows from vertex i to vertex j and the set of vertices of $\mathrm{E}(\Lambda)$ corresponds to the nonisomorphic graded simple $\mathrm{E}(\Lambda)$ modules generated in degree 0. Assume that we have a nonsplit exact sequence of Λ-modules $0\to S_j\to X\to S_i\to 0$. (For the graded case we must consider a nonsplit exact sequence in $gr\Lambda$.) By applying $\mathrm{Hom}_\Lambda(-,\Lambda/\mathbf{r})$ ($\mathrm{Hom}_{gr\Lambda}(-,\Lambda_0)$ in the graded case) we obtain the long exact sequence of Ext-spaces:

$$\dots\mathrm{Hom}_\Lambda(S_j,\Lambda/\mathbf{r}) \xrightarrow{\delta} \mathrm{Ext}^1_\Lambda(S_i,\Lambda/\mathbf{r}) \longrightarrow \mathrm{Ext}^1_\Lambda(X,\Lambda/\mathbf{r}) \longrightarrow$$

$$\longrightarrow \mathrm{Ext}^1_\Lambda(S_j,\Lambda/\mathbf{r}) \xrightarrow{\delta} \mathrm{Ext}^2_\Lambda(S_i,\Lambda/\mathbf{r}) \longrightarrow \cdots$$

where the maps δ denote the connecting homomorphisms. Using these connecting homomorphisms we obtain a nonzero map in $Gr\mathrm{E}(\Lambda)$ from $E(S_j)$ to $\mathbf{r}E(S_i)(1)$ which induces a graded map $E(S_j) \to \mathbf{r}E(S_i)(1)/\mathbf{r}^2E(S_i)(1)$ therefore proving that the quiver of Λ is a full subquiver of the quiver of $\mathrm{E}(\Lambda)$. $\square$

We want next to study under which circumstances we have an equality between the quiver of Λ, $Q(\Lambda)$ and that of its Ext-algebra $Q(\mathrm{E}(\Lambda))$. We have the following:

Proposition 3.2. *$Q(\Lambda) = Q(\mathrm{E}(\Lambda))$ if and only if the algebra $\mathrm{E}(\Lambda)$ is generated in degrees 0 and 1.*

Proof. Clearly $Q(\Lambda) = Q(\mathrm{E}(\Lambda))$ if and only if $\mathrm{E}(\Lambda)_1 = \mathrm{Ext}^1_\Lambda(\Lambda/\mathbf{r}, \Lambda/\mathbf{r}) = \mathbf{r}_{\mathrm{E}(\Lambda)}/\mathbf{r}^2_{\mathrm{E}(\Lambda)}$ where $\mathbf{r}_{\mathrm{E}(\Lambda)}$ denotes the radical of the Ext-algebra. We need the following:

Lemma 3.3. *Let $S = \oplus_{i\geq 0} S_i$ be a graded K-algebra and assume that $S_1 = \operatorname{rad} S$. Then S is generated in degrees 0 and 1:*

Proof. We have $\operatorname{rad} S = 0 \oplus S_1 \oplus S_2 \oplus \dots$ and $\operatorname{rad}^2 S = 0 \oplus 0 \oplus S_1^2 \oplus (S_1S_2 + S_2S_1 + S_1^3) \oplus \dots$. Since $S_1 = \operatorname{rad} S/\operatorname{rad}^2 S$ we must have $S_2 = S_1^2$. Then $S_1S_2 + S_2S_1 + S_1^3 = S_3$ and since $S_2 = S_1^2$ we get $S_3 = S_1^3$, and the rest follows by induction. $\square$

The proposition follows now immediately. $\square$

Definition 3.4. The *shriek* algebra $\Lambda^!$ of a K-algebra Λ is the subalgebra of $\mathrm{E}(\Lambda)$ generated by $\mathrm{E}(\Lambda)_0$ and $\mathrm{E}(\Lambda)_1$.

Therefore, another way to reformulate proposition 3.3 is by saying that $Q(\Lambda) = Q(\mathrm{E}(\Lambda))$ if and only if $\Lambda^! = \mathrm{E}(\Lambda)$. We turn our attention now to another way of characterizing K-algebras Λ with the property that $Q(\Lambda) = Q(\mathrm{E}(\Lambda))$. We will assume from now on that each K-algebra Λ is graded $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \dots$ and generated in degrees 0 and 1, that is $\Lambda_i\Lambda_j = \Lambda_{i+j}$ for all i, j and $\Lambda_0 = K \times \dots \times K$.

Definition 3.5. Let $\Lambda = \oplus_{i\geq 0}\Lambda_i$ be a graded algebra as above and let $M = M_0 \oplus M_1 \oplus \dots$ be a graded Λ-module. We say that M has a *linear resolution* if there exists a graded projective resolution of M:

$$\dots \xrightarrow{\delta_{i+1}} \mathcal{P}_i \to \dots \to \mathcal{P}_1 \xrightarrow{\delta_1} \mathcal{P}_0 \to M \to 0$$

where, for each i, $\mathcal{P}_i$ is graded projective and generated in degree i.

Remarks. (1) If M has a linear resolution then M is generated in degree 0. (2) Every linear resolution is automatically a minimal (graded) projective resolution.

Definition 3.6. Modules having linear resolutions are also called *Koszul* modules. A graded K-algebra Λ generated in degrees 0 and 1, is a *Koszul* algebra if $\Lambda_0 = \Lambda/\mathbf{r}$ has a linear resolution.

We have the following result relating modules with linear resolutions to certain modules over the Ext-algebra.

Proposition 3.7. *The following are equivalent for a graded Λ-module M generated in degree 0.*

1. *M is a Koszul module.*
2. *For each $j \geq 0$, $\mathrm{Ext}^1_\Lambda(\Lambda/\mathbf{r}, \Lambda/\mathbf{r}) \cdot \mathrm{Ext}^j_\Lambda(M, \Lambda/\mathbf{r}) = \mathrm{Ext}^{j+1}_\Lambda(M, \Lambda/\mathbf{r})$ (that is, the module $E(M)$ is generated in degree 0 as a graded $\mathrm{E}(\Lambda)$-module).*

Proof. It is clear that (2) is equivalent to $E(M)$ being generated in degree 0 since for instance $\mathrm{E}(\Lambda)_1E(M)_0 = E(M)_1$ and then we have $E(M)_2 \supseteq \mathrm{E}(\Lambda)_2E(M)_0 \supseteq \mathrm{E}(\Lambda)_1\mathrm{E}(\Lambda)_1E(M)_0 = \mathrm{E}(\Lambda)_1E(M)_1 = E(M)_2$ etc. We prove here only (2) $\Rightarrow$ (1), see [GM1] for instance, for the complete proof. Let $\dots \to \mathcal{P}_n \to \dots \to \mathcal{P}_1 \to \mathcal{P}_0 \to M \to 0$ be a minimal graded projective resolution of M. Since M is generated in degree 0, $\mathcal{P}_0$ is also generated in degree 0. We have an exact sequence in $gr\Lambda : 0 \to \Omega_1 \to \mathcal{P}_0 \to M \to 0$. The implication will follow by induction if we prove that Ω_1 is generated in degree 1. We have the following commutative diagram with exact rows in $gr\Lambda$:

$$\begin{array}{rccccccccc}
 & 0 & \longrightarrow & \Omega_1 & \longrightarrow & \mathcal{P}_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & & \downarrow & & \downarrow & & \| & & \\
\eta: & 0 & \longrightarrow & \Omega_1/\mathbf{r}\Omega_1 & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 \\
 & & & \| & & \downarrow & & \downarrow{\scriptstyle f_i} & & \\
\xi_i: & 0 & \longrightarrow & \Omega_1/\mathbf{r}\Omega_1 & \longrightarrow & X_i & \longrightarrow & \Lambda/\mathbf{r} & \longrightarrow & 0
\end{array}$$

where $\eta \in \operatorname{Ext}^1_\Lambda(M, \Omega_1/\mathbf{r}\Omega_1)$ and $\eta = \sum \xi_i f_i$ by assumption where $f_i \in \operatorname{Hom}(M, \Lambda/\mathbf{r})$ and $\xi_i \in \operatorname{Ext}^1_\Lambda(\Lambda/\mathbf{r}, \Omega_1/\mathbf{r}\Omega_1)$ and $\Omega_1/\mathbf{r}\Omega_1$ is a direct sum of summands of $\Lambda/\mathbf{r}$. Since M is generated in degree 0, $\operatorname{Im} f_i$ are all generated in degree 0 and also each X_i is generated in degree 0. Thus $\Omega_1/\mathbf{r}\Omega_1$ has all its summands in degree 1 and then Ω_1 is generated in degree 1. □

We turn now our attention to studying Koszul algebras in terms of generators and relations.

Definition 3.8. Let Q be a finite quiver and let I be a two-sided ideal of the path algebra KQ. I is *quadratic* if it is generated by elements of the form $\sum \alpha_i p_i$ where $\alpha_i \in K$ and p_i are all paths in Q of length 2. An algebra Λ is *quadratic* if Λ can be written as KQ/I for some quiver Q and some quadratic ideal I of KQ.

Note that if Λ is a quadratic algebra, then Λ is also a graded K-algebra generated in degrees 0 and 1. We shall use the following formulae of Butler that can be found in [Bo], where J denotes the ideal of KQ generated by the arrows of Q, $I \subseteq J^2$ and $\Lambda = KQ/I$:

(3.9)

$$\operatorname{Tor}^\Lambda_{2n}(\Lambda/\mathbf{r}, \Lambda/\mathbf{r}) \cong \frac{I^n \cap JI^{n-1}J}{JI^n + I^nJ} \qquad n \geq 1$$

$$\operatorname{Tor}^\Lambda_{2n+1}(\Lambda/\mathbf{r}, \Lambda/\mathbf{r}) \cong \frac{JI^n \cap I^nJ}{I^{n+1} + JI^nJ} \qquad n \geq 0$$

We have the following well known result. (See for instance [BGS], [GM1]).

Proposition 3.10. *Let Λ be a Koszul algebra. There Λ is quadratic.*

Proof. Let $\cdots \to \mathcal{P}_n \to \cdots \to \mathcal{P}_0 \to \Lambda/\mathbf{r} \to 0$ be a minimal graded resolution of $\Lambda/\mathbf{r}$ over Λ. Just as we had for $\operatorname{Ext}^n_\Lambda(\Lambda/\mathbf{r}, \Lambda/\mathbf{r})$ in the first section, we see that for each $n \geq 0$ we may identify $\mathcal{P}_n/\mathbf{r}\mathcal{P}_n$ with $\operatorname{Tor}^\Lambda_n(\Lambda/\mathbf{r}, \Lambda/\mathbf{r})$. Since $\mathcal{P}_2$ is generated in degree 2, using (3.9) we have that $I/IJ + JI$ is generated by linear combinations of paths of length 2. But I is a homogeneous ideal of KQ and it is easy to see that in this case any minimal system of generators of I lies outside $IJ + JI$. This implies that I is a quadratic ideal. □

Not every quadratic algebra generated in degrees 0 and 1 is Koszul, however if Λ is a quadratic monomial algebra, then Λ is Koszul ([GZ]). This converse is also true for global dimension two ([GM1]):

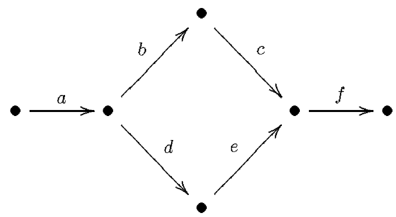
Proposition 3.11. *Let Λ be a quadratic global dimension 2 algebra. Then Λ is a Koszul algebra.*

Proof. The result follows immediately using the proof of 3.10. Indeed if we have a minimal graded resolution $0 \to \mathcal{P}_2 \to \mathcal{P}_1 \to \mathcal{P}_0 \to \Lambda/\mathbf{r} \to 0$ then $\mathcal{P}_0$ and $\mathcal{P}_1$ must be generated in degrees 0 and 1 respectively, and, since $\mathcal{P}_2/\mathbf{r}\mathcal{P}_2 \cong I/IJ + JI$ and I is quadratic, it follows that each summand of $\mathcal{P}_2$ is generated in degree 2 so that Λ is Koszul. □

It is not known for which algebras the notions of Koszul and quadratic coincide.

Example 3.12. Let Λ be given by the quiver Q subject to the relation ideal I where:

Q :

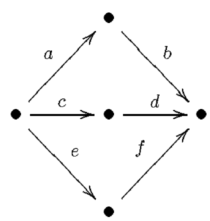


$I =< ba, fe, cb - ed >$. Then $\Lambda = KQ/I$ is quadratic but not Koszul. Note that Λ has global dimension three.

It is well known how to construct the shriek algebra of a quadratic algebra ([BGS], [GM1], [S]): if $\Lambda = KQ/I$ is a quadratic algebra, let $V \subset KQ$ be the subspace spanned by all the paths of length 2. We have the usual bilinear form $<,>: V \times V^* \to K$ given by $< p, q^* >= q^*(p)$ for p, q in V. Let $I_2 = I \cap (KQ)_2$, and let $I_2^\perp = \{x \in V | x^*(\rho) = 0 \text{ for all } \rho \text{ in } I_2\}$. Then $\Lambda^! = KQ/ < I_2^\perp >$. In particular $\Lambda^!$ is also quadratic. In this way if $\Lambda = KQ/I$ is a Koszul algebra, it is very easy to determine $\mathrm{E}(\Lambda)$ in terms of its quiver and its relations.

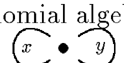
Example 3.13. (i) Let $\Lambda = KQ/I$ where

Q :



where $I =< ba - dc >$. The global dimension of Λ is 2, so Λ is a Koszul algebra. We obtain $\mathrm{E}(\Lambda) = \Lambda^! = KQ/ < fe, ba + dc >$.

(ii) If $\Lambda = KQ$ the path algebra of a finite quiver, then Λ is obviously quadratic so $\mathrm{E}(\Lambda) = \Lambda^! = KQ/J^2$ where J is the two-sided ideal generated by the arrows. Similarly $E(KQ/J^2) = KQ$ so in this case we see that $E(E(KQ)) = KQ$ and $E(E(KQ/J^2)) = KQ/J^2$.

(iii) Let $\Lambda = K[x, y]$ be the polynomial algebra in two variables. We can realize Λ as a quotient of the path algebra KQ where Q is 

. Then $KQ = K < x, y >$ is the free algebra in two noncommuting variables and $\Lambda = K < x, y > / < xy - yx >$. In this case $\mathrm{E}(\Lambda) = K < x, y > / < x^2, y^2, xy + yx >$ is the exterior algebra.

Another way to construct Koszul algebras from known Koszul algebras is using tensor products. Note that if Λ and Λ' are graded K-algebras, then $\Lambda \underset{K}{\otimes} \Lambda'$ is also a graded K-algebra, the grading being induced by the total grading. We have the following well-known result, see [GM1] for instance:

Proposition 3.14. *Let Λ and Λ' be two Koszul algebras. Then $\Lambda \underset{K}{\otimes} \Lambda'$ is a Koszul algebra.*

In particular, if Λ is Koszul, its enveloping algebra $\Lambda^e = \Lambda \underset{K}{\otimes} \Lambda^{op}$ is also Koszul. Also, a polynomial ring $K[x_1, \ldots, x_n]$ is a Koszul algebra since $K[x_1, \ldots, x_n] \cong K[x_1] \underset{K}{\otimes} \ldots \underset{K}{\otimes} K[x_n]$.

Let $\Lambda = KQ/I$ be a graded algebra generated in degrees 0 and 1. Let Q_0 denote the set of vertices of Q and let Q_1 denote the set of arrows of Q. We can view Λ as a graded Λ^e module as follows: for each vertex v of Q, let Λv $(v\Lambda)$ denote the graded indecomposable left projective Λ-module (right projective) generated in degree 0 corresponding to the vertex v, and let $\mathcal{P}_0 = \coprod_{v \in Q_0} \Lambda v \underset{K}{\otimes} v\Lambda$. Clearly $\mathcal{P}_0$ is a graded projective Λ^e-module generated in degree 0. For each arrow a of Q, let $o(a)$ denote the origin of a and let $t(a)$ denote its terminus. In this notation, let $\mathcal{P}_1 = \coprod_{a \in Q_1} \Lambda o(a) \underset{K}{\otimes} t(a)\Lambda$. By viewing $\Lambda o(a)$ as a graded left Λ-module generated in degree 0 and $t(a)\Lambda$ as a graded right Λ-module generated in degree 1 we see that $\mathcal{P}_1$ is a graded Λ^e-projective module generated in degree 1. Furthermore, the Λ^e-homorphism $\phi : \mathcal{P}_1 \to \mathcal{P}_0$ given by $\phi(o(a) \otimes t(a)) = a \otimes t(a) - o(a) \otimes a$ is clearly a degree 0 morphism and, since Λ is the cokernel of ϕ, it follows that Λ can be viewed as a graded Λ^e-module generated in degree 0. Note that $\mathcal{P}_1 \xrightarrow{\ \phi\ } \mathcal{P}_0 \to \Lambda \to 0$ is a minimal graded projective presentation of Λ over Λ^e ([Ba]). The following characterization of Koszul algebras is due to the author, Ed Green and Roberto Martinez-Villa ([GMZ]). One direction has been independently obtained by Skoldberg and Johansson and announced at the ICRA conference in Geiranger in 1996. The proof presented here is more elegant than the original one, and is due to Eduardo Marcos.

Theorem 3.15. *Let Λ be a graded K algebra generated in degrees 0 and 1. Then Λ is a Koszul algebra if and only if viewed as a Λ^e module, Λ has a linear resolution.*

Proof. The proof will be done in a few steps. Observe that if $\mathcal{P} = \Lambda v \underset{K}{\otimes} w\Lambda$ is an indecomposable projective Λ^e-module, and if M is a Λ-module then $\mathcal{P} \underset{\Lambda}{\otimes} M = (\Lambda v)^{\dim wM}$ as a Λ-module, since $\Lambda v \underset{K}{\otimes} w\Lambda \underset{\Lambda}{\otimes} M \cong \Lambda v \underset{K}{\otimes} wM$. In particular, if $M = S$ is a simple Λ-module, then, as Λ-modules we have $\mathcal{P} \underset{\Lambda}{\otimes} S \simeq \Lambda v$ if $wS \neq 0$ (i.e. if $S \simeq \Lambda w/\mathbf{r}w$) and $\mathcal{P} \underset{\Lambda}{\otimes} S = 0$ otherwise. Our next step is the following

Lemma 3.16. *Let $f : \mathcal{P} \to \mathcal{Q}$ be a homomorphism of finitely generated Λ^e-projective modules. Then $\operatorname{Im} f \subseteq \mathbf{r}_{\Lambda^e}\mathcal{Q}$ if and only if for each simple Λ-module S, we have $\operatorname{Im}(f \underset{\Lambda}{\otimes} 1_S) \subseteq \mathbf{r}_\Lambda(\mathcal{Q} \underset{\Lambda}{\otimes} S)$.*

Proof. It is clear that we may assume that $\mathcal{Q}$ is an indecomposable Λ^e-module, so $\mathcal{Q} = \Lambda v \underset{K}{\otimes} w\Lambda$. Assume that f is onto, so f is a splittable epimorphism and by tensoring it with any Λ-module we get an epimorphism of Λ-modules. In particular, if we choose $S = \Lambda w/\mathbf{r}w$, we get an epimorphism $f \otimes 1_S : \mathcal{P} \underset{\Lambda}{\otimes} S \to \Lambda v$ so one of the implications is proved. For the other direction, assume that $\operatorname{Im} f \subseteq \mathbf{r}_{\Lambda^e}\mathcal{Q}$. Since for each simple Λ-module $T \ncong \Lambda w/\mathbf{r}w$ we have $\mathcal{Q} \underset{\Lambda}{\otimes} T = 0$, it is enough to prove that if $S = \Lambda w/\mathbf{r}w$, then $\operatorname{Im}(f \otimes 1_S) \subseteq \Lambda v$. We have the following commutative diagram of Λ-modules:

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\ f\ } & \mathcal{Q} \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ \mathcal{P} \underset{\Lambda}{\otimes} S & \xrightarrow{\ f \otimes 1_S\ } & \mathcal{Q} \underset{\Lambda}{\otimes} S \end{array}$$

where α and β are the splittable Λ-epimorphisms given by the split exact sequences in mod Λ: $0 \to \Lambda v \underset{K}{\otimes} w\mathbf{r}_\Lambda \to \Lambda v \underset{K}{\otimes} w\Lambda \xrightarrow{\ \beta\ } \Lambda v \to 0$ for β, (and similarly defined for α). But the map β has the property that $\beta^{-1}(v) = v \otimes w + \Lambda v \otimes w\mathbf{r}$ so each element in the preimage of v is a Λ^e-generator for the module $\mathcal{Q} = \Lambda v \underset{K}{\otimes} w\Lambda$. If $f \otimes 1_S$ is onto, then $\beta \circ f$ is onto and $\beta^{-1}(v) \cap \operatorname{Im} f \neq 0$. But this implies that $\operatorname{Im} f$ contains a Λ^e-generator of the cyclic module $\mathcal{Q}$ thus f is onto and the lemma is proved. $\square$

We may use the lemma to finish the proof of the theorem. Let

$$(\mathcal{P}): \qquad \cdots \to \mathcal{P}_n \to \cdots \to \mathcal{P}_0 \to \Lambda \to 0$$

be a graded Λ^e-resolution of Λ. We have seen that $(\mathcal{P})$ is minimal if and only if the induced resolution

$$(\mathcal{P} \underset{\Lambda}{\otimes} \Lambda/\mathbf{r}): \qquad \cdots \to \mathcal{P}_n \underset{\Lambda}{\otimes} \Lambda/\mathbf{r} \to \cdots \to \mathcal{P}_0 \underset{\Lambda}{\otimes} \Lambda/\mathbf{r} \to \Lambda/\mathbf{r} \to 0$$

is a minimal resolution of $\Lambda/\mathbf{r}$. By our earlier remarks, $\mathcal{P}_n$ is generated in degree n if and only if the projective Λ-module $\mathcal{P}_n \underset{\Lambda}{\otimes} \Lambda/\mathbf{r}$ is generated in degree n. The proof is now complete. □

Since Λ being a Koszul Λ^e-module is in fact a condition for Λ-bimodules we have as a consequence a different proof of the following well-known result ([BGS], [GM1]):

Corollary 3.17. *Let Λ be a graded K-algebra generated in degrees 0 and 1. Then Λ is Koszul if and only if Λ^{op} is Koszul.*

We summarize our results in the following:

Theorem 3.18. *The following statements are equivalent for a graded algebra Λ generated in degrees 0 and 1:*

(i) Λ is a Koszul algebra.
(ii) Λ^{op} is a Koszul algebra.
(iii) $\mathrm{E}(\Lambda)$ is generated in degrees 0 and 1.
(iv) Λ is a Koszul module over Λ^e.
(v) $Q(\Lambda) = Q(\mathrm{E}(\Lambda))$.

4. Koszul duality. Modules with linear presentations.

We want now to study the relationship between Koszul modules over a Koszul algebra and the Koszul modules over the corresponding Ext-algebra.

Definition 4.1. Let Λ be a graded algebra generated in degrees 0 and 1. We denote by $\mathcal{K}_\Lambda$ the full subcategory of $gr\Lambda$ consisting of all the Koszul modules generated in degree 0.

Thus, Λ is a Koszul algebra if and only if $\Lambda/\mathbf{r} \in \mathcal{K}_\Lambda$, or equivalently, if and only if $\Lambda \in \mathcal{K}_{\Lambda^e}$. We will need the following result([GM1]):

Lemma 4.2. *Let Λ be a Koszul algebra and let $\mathbf{r}$ be its (graded) radical. Then for each $i \geq 1$, $\mathbf{r}^i$ is a Koszul module. If M is a Koszul module, then for each $i \geq 1$, $\mathbf{r}^i M$ is a Koszul module.*

We use this lemma to prove the following result ([GM1], [ADL2]).

Proposition 4.3. *Let Λ be a Koszul algebra and let $M \in \mathcal{K}_\Lambda$. Then, for each $i \geq 1$, the induced map $\mathrm{Ext}^i_\Lambda(M/\mathbf{r}M, \Lambda/\mathbf{r}) \to \mathrm{Ext}^i_\Lambda(M, \Lambda/\mathbf{r})$ is onto.*

Proof. We have the following commutative diagram with exact rows and columns, where P denotes the graded projective cover of M in $gr\Lambda$:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \Omega_1(M) & \longrightarrow & \Omega_1(M/\mathbf{r}M) & \longrightarrow & \mathbf{r}M \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & P & = & P & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbf{r}M & \longrightarrow & M & \longrightarrow & M/\mathbf{r}M & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since Λ is Koszul, the terms of the top sequence are Koszul modules and they are all generated in degree 1. This tells us that a minimal graded resolution of $\Omega_1(M/\mathbf{r}M)$ can be constructed as the direct sum of two minimal graded resolutions for $\Omega_1(M)$ and for $\mathbf{r}M$, and this implies the proposition. □

Let Λ be a Koszul algebra and let $M \in \mathcal{K}_\Lambda$ be a nonsemisimple module. In $gr\Lambda$ we have the exact sequence $0 \to \mathbf{r}M \to M \to M/\mathbf{r}M \to 0$, and, by applying the functor $\mathrm{Hom}_\Lambda(-, \Lambda/\mathbf{r})$ we obtain using 4.3 the following exact sequence in $gr\mathrm{E}(\Lambda)$: $0 \to E(\mathbf{r}M)(1) \to E(M/\mathbf{r}M) \to E(M) \to 0$, that is $\Omega^1_{\mathrm{E}(\Lambda)}(E(M)) = E(\mathbf{r}M)(1)$ where $E(\mathbf{r}M)(1)$ denotes the shift of $E(\mathbf{r}M)$ as defined in the first section.

Continuing we obtain the following ([GM1], [ADL2], [S]):

Theorem 4.4. *Let Λ be a Koszul algebra and let $M \in \mathcal{K}_\Lambda$. Then*
(i) $E(M) \in \mathcal{K}_{\mathrm{E}(\Lambda)}$ and the resolution:

$$\cdots \to E(\mathbf{r}^i M/\mathbf{r}^{i+1}M)(i) \to E(\mathbf{r}^{i-1}M/\mathbf{r}^i M)(i-1) \to \cdots \to E(M/\mathbf{r}M) \to E(M) \to 0$$

is a linear $\mathrm{E}(\Lambda)$-projective resolution of $E(M)$.
(ii) $\mathrm{E}(\Lambda)$ is a Koszul algebra
(iii) $\mathbf{r}^n M = 0$ for some $n \geq 1$ if and only if $pd_{\mathrm{E}(\Lambda)}E(M) \leq n$.

Proof. Part (i) follows from the preceding discussion by applying $\mathrm{Hom}_\Lambda(-, \Lambda/\mathbf{r})$ to the sequences $0 \to \mathbf{r}^{i+1}M \to \mathbf{r}^i M \to \mathbf{r}^i M/\mathbf{r}^{i+1}M \to 0$; part (ii) follows immediately from (i) by letting $M = \Lambda$ and using 3.1. (iii) is a trivial consequence of (i). □

Theorem 4.5. *(*[BGS], [GM1], [S], [ADL2]*) Let Λ be a Koszul algebra. Then:*
There exist inverse dualities $\mathcal{K}_\Lambda \underset{G}{\overset{F}{\leftrightarrows}} \mathcal{K}_{\mathrm{E}(\Lambda)}$ given by $F(M) = \underset{n\geq 0}{\oplus} \mathrm{Ext}^n_\Lambda(M, \Lambda/\mathbf{r})$ for each $M \in \mathcal{K}_\Lambda$, and $G(M) = \underset{n\geq 0}{\oplus} \mathrm{Ext}^n_{\mathrm{E}(\Lambda)}(N, \mathrm{E}(\Lambda)_0)$ for each $N \in \mathcal{K}_{\mathrm{E}(\Lambda)}$.

Theorem 4.6. *(see* [GM1] *for instance). Let Λ be a graded algebra generated in degrees 0 and 1. Then Λ is a Koszul algebra if and only if Λ is isomorphic to $E(\mathrm{E}(\Lambda))$ as graded algebras.*

Proof. If $\Lambda \approx E(\mathrm{E}(\Lambda))$ as graded algebras, then $\mathrm{E}(\Lambda)$ is Koszul by 3.18. Using the previous result, $E(\mathrm{E}(\Lambda))$ and thus Λ are also Koszul algebras. If we start with a Koszul algebra Λ, then $\Lambda^! = \mathrm{E}(\Lambda)$ and since $\Lambda \approx \Lambda^{!!}$ as graded algebras, the result follows immediately. □

We try now to describe some properties of $\mathcal{K}_\Lambda$ in some special situations.

Definition 4.7. Let Λ be a graded algebra generated in degrees 0 and 1. We denote by $gr_0\Lambda$ the full subcategory of $gr\Lambda$ consisting of the modules generated in degree 0. A module $M \in gr_0\Lambda$ has a *linear presentation* if there exists a graded projective presentation $\mathcal{P}_1 \to \mathcal{P}_0 \to M \to 0$ with $\mathcal{P}_0$ generated in degree 0, and $\mathcal{P}_1$ generated in degree 1. We denote by $\mathcal{L}(\Lambda)$ the full subcategory of $gr_0\Lambda$ consisting of the modules having a linear presentation.

We obviously have $\mathcal{K}_\Lambda \subseteq \mathcal{L}(\Lambda) \subseteq gr_0\Lambda$. Since as a Λ-module, Λ is generated in degree 0 we clearly have $\Lambda/\mathbf{r} \in \mathcal{L}(\Lambda)$. We have the following equivalence of categories ([GMRSZ]).

Proposition 4.8. *There is an equivalence of categories $\mathcal{L}(\Lambda) \approx \mathcal{L}(\Lambda/\mathbf{r}^2)$.*

Proof. One direction is the functor $F : \mathcal{L}(\Lambda) \to \mathcal{L}(\Lambda/\mathbf{r}^2)$, $F(M) = M/\mathbf{r}^2M$ which turns out to be exact, and there is an obvious inverse to F which is also exact. □

Corollary 4.9. *([GMRSZ]) Let Λ be a graded algebra generated in degrees 0 and 1. Then $\mathcal{L}(\Lambda)$ has almost split sequences.*

Proof. It was proven in [GMRSZ] that $gr_0\Gamma$ has almost split sequences for every algebra Γ generated in degrees 0 and 1. In our case we have exact equivalences $\mathcal{L}(\Lambda) \approx \mathcal{L}(\Lambda/\mathbf{r}^2) = gr_0(\Lambda/\mathbf{r}^2)$. □

We now have the following.

Theorem 4.10. *([GMRSZ]) Let Λ be a monomial Koszul algebra. Then $\mathcal{K}_\Lambda = \mathcal{L}(\Lambda)$. In particular $\mathcal{K}_\Lambda$ has almost split sequences.*

It is not known for which algebras the categories of modules with linear presentations and of the Koszul modules generated in degree 0 coincide. We have the following ([GMZ]).

Theorem 4.11. *Let Λ be a Koszul algebra such that $\mathcal{L}(\Lambda) = \mathcal{K}_\Lambda$. Then $\mathcal{L}(\mathrm{E}(\Lambda)) = \mathcal{K}_{\mathrm{E}(\Lambda)}$.*

Proof. It was shown in [GMRSZ] that there is an equivalence of categories $H : \mathcal{L}(\Lambda) \to gr_0(\mathrm{E}(\Lambda)/\mathbf{r}^2\mathrm{E}(\Lambda)) = \mathcal{L}(\mathrm{E}(\Lambda)/\mathbf{r}^2\mathrm{E}(\Lambda))$ given by $H(M) = \mathrm{Ext}^0_\Lambda(M, \Lambda/\mathbf{r}) \oplus \mathrm{Ext}^1_\Lambda(M, \Lambda/\mathbf{r})$. We have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{L}(\Lambda) = \mathcal{K}_\Lambda & \xrightarrow{\;F\;} & \mathcal{K}_{\mathrm{E}(\Lambda)} \\ {\scriptstyle H}\big\downarrow & & \big\downarrow{\scriptstyle i} \\ \mathcal{L}(\mathrm{E}(\Lambda)/\mathbf{r}^2\mathrm{E}(\Lambda)) & \xleftarrow{\;G\;} & \mathcal{L}(\mathrm{E}(\Lambda)) \end{array}$$

where i is the inclusion functor, F is the usual Koszul duality G is the equivalence $G(X) = X/\mathbf{r}^2X$, and H is as above. It follows that i is also an equivalence so it is dense, thus we get $\mathcal{K}_{\mathrm{E}(\Lambda)} = \mathcal{L}(\mathrm{E}(\Lambda))$. □

We would like to conclude by observing that in general the category $\mathcal{K}_\Lambda$ does not have almost split sequences. In fact we have the following

Proposition 4.12. *([GMRSZ]). Let Λ be a selfinjective Koszul algebra such that $\mathbf{r}^3 \neq 0$. There exists an indecomposable Koszul module $N \in \mathcal{K}_\Lambda$ for which there is no almost split sequence of the form $0 \to N \to B \to M \to 0$ in $\mathcal{K}_\Lambda$.*

It is interesting to note that in the Koszul selfinjective case, $\mathcal{K}_\Lambda$ has right almost split sequences in the following sense:

Proposition 4.13. *([GMRSZ]) Let Λ be a selfinjective Koszul algebra and let $M \in \mathcal{K}_\Lambda$ be an indecomposable nonprojective module. There exists an almost split sequence $0 \to A \to B \to M \to 0$ in $\mathcal{K}_\Lambda$.*

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