# Hochschild cohomology and stable equivalences

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# Introduction

$K$ is an algebraically closed field.
All algebras are (associative, unitary) $K$-algebras which are finite dimensional and indecomposable.
Many authors, such as Skowroński, Bocian, Holm, Białkowski, Zimmermann... are interested in tame finite dimensional algebras, in particular those that are selfinjective, for instance

- blocks of group algebras,
- Hopf algebras,
- Brauer graph algebras,
- Erdmann's algebras...

Their aim is to classify them, up to equivalences of categories, such as Morita equivalence, derived equivalence, stable equivalence.
I am going to talk about some invariants of equivalences of categories related to Hochschild cohomology and give some applications.

1. Hochschild cohomology

2. Equivalences of categories

3. Invariants associated to the first Hochschild cohomology groups

4. Application to the classification of symmetric algebras of dihedral, semi-dihedral and quaternion type

5. Application to the classification of generalisations of Nakayama algebras

# Hochschild cohomology

## Hochschild complex

Let $A$ be a finite-dimensional $K$-algebra.
The Hochschild cohomology of $A$ is the cohomology of the complex

$$0 \to \mathrm{Hom}_K(K, A) \xrightarrow{d_0} \mathrm{Hom}_K(A, A) \xrightarrow{d_1} \mathrm{Hom}_K(A \otimes A, A) \to \cdots$$

$$\cdots \to \mathrm{Hom}_K(A^{\otimes n}, A) \xrightarrow{d_n} \mathrm{Hom}_K(A^{\otimes(n+1)}, A) \xrightarrow{d_{n+1}} \cdots$$

$$\begin{aligned} d_n(f)(a_1 \otimes \cdots \otimes a_n \otimes a_{n+1}) = {} & a_1 f(a_2 \otimes \cdots \otimes a_n) \\ & + \sum_{i=1}^{n} (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) \\ & + (-1)^{n+1} f(a_1 \otimes \cdots \otimes a_n) a_{n+1}, \end{aligned}$$

$$\mathrm{HH}^n(A) = \mathrm{Ker}(d_n)/\,\mathrm{Im}(d_{n-1}) \text{ and } \mathrm{HH}^*(A) = \bigoplus_{n \in \mathbb{N}} \mathrm{HH}^n(A).$$

# Hochschild cohomology

We will be interested in $\mathrm{HH}^0(A)$ and $\mathrm{HH}^1(A)$.
The first differentials:

$$0 \to \mathrm{Hom}_K(K, A) \xrightarrow{d_0} \mathrm{Hom}_K(A, A) \xrightarrow{d_1} \mathrm{Hom}_K(A \otimes A, A) \to \dots$$

- If $f \in \mathrm{Hom}_K(K, A)$, then $d_0(f)(a) = af(1) - f(1)a$ for all $a \in A$.
- If $g \in \mathrm{Hom}_K(A, A)$, then $d_1(g)(a \otimes b) = ag(b) - g(ab) + g(a)b$.

$$\begin{array}{ccc} \mathrm{Hom}_K(K, A) & \xrightarrow{\cong} & A \\ f & \longmapsto & f(1) \end{array}$$

Then $\mathrm{HH}^0(A) = \mathrm{Ker}(d_0)$ identifies with $\{z \in A \mid \forall a \in A,\ az = za\}$, that is, the centre $Z(A)$ of $A$.

# Hochschild cohomology

$$\operatorname{Ker} d_1 = \{g \in \operatorname{End}_K(A) \mid g(ab) = ag(b) + g(a)b\}$$

The elements of $\operatorname{Ker} d_1$ are the $K$-derivations of $A$.

$\operatorname{Im} d_0$ identifies with the derivations of the form $D_c : a \mapsto ac - ca$, called inner derivations.

So $\mathrm{HH}^1(A)$ is the quotient of the set of derivations of $A$ by the inner derivations of $A$.

## Structure of derivations

$$D, D' \text{ derivations of } A \;\Rightarrow\; D \circ D' - D' \circ D \text{ derivation of } A.$$

Indeed,

$$\begin{aligned} D \circ D'(ab) &= D(aD'(b) + D'(a)b) \\ &= aD \circ D'(b) + D(a)D'(b) + D'(a)D(b) + D \circ D'(a)b \\ D' \circ D(ab) &= D'(aD(b) + D(a)b) \\ &= aD' \circ D(b) + D'(a)D(b) + D(a)D'(b) + D' \circ D(a)b \end{aligned}$$

and the difference is

$$\begin{aligned} &(D \circ D' - D' \circ D)(ab) \\ &\qquad = a\left(D \circ D' - D' \circ D\right)(b) + \left(D \circ D' - D' \circ D\right)(a)b. \end{aligned}$$

This derivation is denoted by $[D, D'] = D \circ D' - D' \circ D$.

# Hochschild cohomology

Moreover, the bracket of a derivation and an inner derivation is an inner derivation:

$$[D, D_c] = D_{D(c)}.$$

Therefore the bracket above induces a bracket on $\mathrm{HH}^1(A)$.
$\mathrm{HH}^1(A)$ endowed with this bracket is a *Lie algebra*, that is, the bracket is bilinear, it satisfies $[D, D] = 0$ for all $D$ and it satisfies the Jacobi identity

$$[D_1, [D_2, D_3]] + [D_2, [D_3, D_1]] + [D_3, [D_1, D_2]] = 0.$$

# Hochschild cohomology

## Structure of Hochschild cohomology

The Hochschild cohomology $\mathrm{HH}^*(A) = \bigoplus_{n\in\mathbb{N}} \mathrm{HH}^n(A)$ is a graded algebra, whose product is the cup-product:

$$\begin{array}{ccc} \mathrm{Hom}_K(A^{\otimes p}, A) \times \mathrm{Hom}_K(A^{\otimes q}, A) & \to & \mathrm{Hom}_K(A^{\otimes (p+q)}, A) \\ (f, g) & \mapsto & f \smile g \end{array}$$

$f \smile g(a_1 \otimes \cdots \otimes a_{p+q}) = f(a_1 \otimes \cdots \otimes a_p)g(a_{p+1} \otimes \cdots \otimes q_{p+q})$ induces

$$\smile : \mathrm{HH}^p(A) \times \mathrm{HH}^q(A) \to \mathrm{HH}^{p+q}(A).$$

The centre $Z(A)$ is then a subalgebra of $\mathrm{HH}^*(A)$.

# Hochschild cohomology

## Structure of Hochschild cohomology

There is also a graded Lie bracket, for a shifted grading:

$$[,] : \mathrm{HH}^p(A) \times \mathrm{HH}^q(A) \to \mathrm{HH}^{p+q-1}(A).$$

The restriction to $\mathrm{HH}^1(A)$ is then the Lie subalgebra structure we had before.

These two structures are compatible and $\mathrm{HH}^*(A)$ is then called a Gerstenhaber algebra.

# Hochschild cohomology

When we want to compute Hochschild cohomology explicitly, the Hochschild complex is too large. Therefore we use other constructions.

Let $\boldsymbol{P}^\bullet : \cdots \to P^{n+1} \xrightarrow{\delta^n} P^n \xrightarrow{\delta^{n-1}} \cdots \to P^2 \xrightarrow{\delta^1} P^1 \xrightarrow{\delta^0} P^0 \xrightarrow{\delta^{-1}} A \to 0$
be a projective $A$-bimodule resolution of $A$,
that is, an exact sequence in which all the $P^n$ are projective $A$-bimodules.

Apply $\mathrm{Hom}_{A-A}(-, A)$ to

$$\cdots \to P^{n+1} \xrightarrow{\delta^n} P^n \xrightarrow{\delta^{n-1}} \cdots \to P^2 \xrightarrow{\delta^1} P^1 \xrightarrow{\delta^0} P^0 \to 0,$$

this gives a complex

$$0 \to \mathrm{Hom}_{A-A}(P^0, A) \xrightarrow{\delta_*^0} \mathrm{Hom}_{A-A}(P^1, A) \xrightarrow{\delta_*^1} \cdots$$
$$\cdots \to \mathrm{Hom}_{A-A}(P^n, A) \xrightarrow{\delta_*^{n+1}} \mathrm{Hom}_{A-A}(P^{n+1}, A) \to \cdots$$

whose cohomology is also the Hochschild cohomology of $A$.

# Hochschild cohomology

Given two projective resolutions $(\boldsymbol{P}^\bullet, \delta^\bullet)$ and $(\boldsymbol{Q}^\bullet, \partial^\bullet)$ of $A$, there always exist comparison morphisms $\boldsymbol{f}^\bullet : \boldsymbol{P}^\bullet \to \boldsymbol{Q}^\bullet$ and $\boldsymbol{g}^\bullet : \boldsymbol{Q}^\bullet \to \boldsymbol{P}^\bullet$ such that $f \circ g$ and $g \circ f$ are quasi-isomorphisms, that is, the cohomology maps they induce are isomorphisms.

If $(\boldsymbol{P}^\bullet, \delta^\bullet)$ is the Bar resolution, then $(\mathrm{Hom}_{A-A}(\boldsymbol{P}^\bullet, A), \delta_*^\bullet)$ identifies with the Hochschild complex.
However, in computations, we often use smaller resolutions, if possible minimal projective resolutions, that is, projective resolutions such that $\mathrm{Im}\, \delta^n \subset \mathrm{Rad}(P^n)$ for all $n \geqslant 0$ and $\mathrm{Im}\, \delta^{-1} \subset \mathrm{Rad}(A)$.

# Hochschild cohomology

There are methods to compute the first few terms of such minimal projective resolutions for general basic algebras (eg. Green-Snashall), but they do not generalise well for higher $n$, except in some cases (monomial algebras – Green-Snashall-Solberg for instance).
There are also methods to compute whole minimal projective resolutions for algebras satisfying some conditions (Chouhy-Solotar for instance).

If we know explicitly comparison morphisms between $(\boldsymbol{P}^\bullet, \delta^\bullet)$ and the Bar resolution, at least for small values of $n$, then we can transport the Lie algebra structure on $\mathrm{HH}^1(A)$ so that it is described in terms of cocycles in $\mathrm{Hom}_{A-A}(P^1, A)$ instead of derivations.

This Lie algebra has been studied in particular by Strametz (using a minimal projective resolution) in the case of a basic monomial algebra. She gives an explicit combinatoric description of the bracket in terms of paths in the quiver.

# Equivalences of categories

We shall use Hochschild cohomology to distinguish algebras up to some equivalences of categories, which I describe briefly here.

# Equivalences of categories

**Morita equivalences.**

## Definition

Two finite dimensional $K$-algebras $A$ and $B$ are Morita equivalent if the categories of left modules $A\text{-}\operatorname{mod}$ and $B\text{-}\operatorname{mod}$ are equivalent.

## Theorem

$A$ and $B$ are Morita equivalent if and only if there exist

- an $A$-$B$-bimodule $M$ and
- a $B$-$A$-bimodule $N$

that are projective as left and right modules and such that

- $M \otimes_B N \cong A$ as $A$-$A$-bimodules
- and $N \otimes_A M \cong B$ as $B$-$B$-bimodules.

The equivalences are then given by

$$M \otimes_B - : B\text{-}\operatorname{mod} \to A\text{-}\operatorname{mod} \qquad \text{and} \qquad N \otimes_A - : A\text{-}\operatorname{mod} \to B\text{-}\operatorname{mod}.$$

# Equivalences of categories

$$A \text{ and } B \text{ Morita equivalent}$$
$$\Downarrow$$
$$\mathrm{HH}^*(A) \cong \mathrm{HH}^*(B)$$

Moreover, if $A$ and $B$ are Morita equivalent,

- the algebras $\mathrm{HH}^0(A)$ and $\mathrm{HH}^0(B)$ are isomorphic,
- the Lie algebras $\mathrm{HH}^1(A)$ and $\mathrm{HH}^1(B)$ are isomorphic.

# Equivalences of categories

**Derived equivalences.**

$\mathcal{K}(A)$: category of complexes of $A$-modules whose homology vanishes for sufficiently large positive and negative degrees.

The bounded derived category $\mathcal{D}^b(A)$ of $A$ is the largest quotient of $\mathcal{K}(A)$ such that quasi-isomorphisms become isomorphisms. This category is naturally a *triangulated category*.

**Definition**

Two algebras $A$ and $B$ are derived equivalent if their bounded derived categories are equivalent as triangulated categories.

# Equivalences of categories

**Theorem (Rickard, 1991)**

$$A \text{ and } B \text{ derived equivalent}$$
$$\Downarrow$$
$$\forall n \in \mathbb{N},\ \mathrm{HH}^n(A) \cong \mathrm{HH}^n(B)$$

Moreover, if $A$ and $B$ are derived equivalent,

- the algebras $\mathrm{HH}^0(A)$ and $\mathrm{HH}^0(B)$ are isomorphic,
- the Lie algebras $\mathrm{HH}^1(A)$ and $\mathrm{HH}^1(B)$ are isomorphic.

Holm in particular has used this invariant in order to classify some of Erdmann's algebras up to derived equivalence.

# Equivalences of categories

**Remark**

The whole of the Gerstenhaber structure of $\mathrm{HH}^*(A)$ is invariant under derived equivalence (Keller 2004).

**Remark**

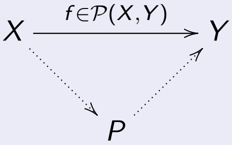
$$
\begin{array}{c}
A \text{ and } B \text{ Morita equivalent} \\
\Downarrow \\
A \text{ and } B \text{ derived equivalent}
\end{array}
$$

# Equivalences of categories

**Stable equivalences.**

### Definition

The **stable category** $A$-$\underline{\text{mod}}$ associated to $A$ has the same objects as $A$- mod and $\underline{\text{Hom}}_A(X, Y) = \text{Hom}_A(X, Y)/\mathcal{P}(X, Y)$ where $\mathcal{P}(X, Y)$ is the space of morphisms of $A$-modules from $X$ to $Y$ that factor through a projective module.




But stable equivalences are not well behaved.

# Equivalences of categories

However, we have the following result.

**Theorem (Keller-Vossieck; Rickard)**

Let $A$ and $B$ be two selfinjective algebras that are derived equivalent. Then they are stably equivalent.
Moreover, there exists

- an $A$-$B$-bimodule $M$ and
- a $B$-$A$-bimodule $N$

that are projective on both sides such that

- $M \otimes_B N \cong A \oplus P$ with $P$ projective and
- $N \otimes_A M \cong B \oplus Q$ with $Q$ projective,

and they induce the equivalence via $M \otimes_B -$ and $N \otimes_A -$.

# Equivalences of categories

### Definition

A stable equivalence between two selfinjective algebras given by two modules as in the theorem is called a stable equivalence of Morita type.

This kind of equivalence has been much studied lately, for instance by Zimmermann, Zhou, König, Liu... and generalised (eg. singular equivalence of Morita type [Zhou-Zimmermann] singular equivalence of Morita type and with level [Wang, Skartsæterhagen...])

# Equivalences of categories

$A$, $B$ selfinjective algebras.

**Theorem (Xi, 2008)**

$$A \text{ and } B \text{ stably equivalent of Morita type}$$
$$\Downarrow$$
$$\forall n \in \mathbb{N}^{>0},\ \mathsf{HH}^n(A) \cong \mathsf{HH}^n(B)$$

This is not necessarily true for $\mathsf{HH}^0(A)$ or for the Lie structure of $\mathsf{HH}^1(A)$ in general, I shall come back to it.

# Invariants associated to the first Hochschild cohomology groups

$\mathrm{HH}^*(A)$ (as a Gerstenhaber algebra) is invariant under derived equivalences, and $\mathrm{HH}^{\geqslant 1}(A)$ is invariant under stable equivalences of Morita type.

Unfortunately, Hochschild cohomology is not easy to compute in general. It is therefore useful to have invariants that are finer and easier to compute.

In order to describe these invariants, I shall need to remind you of symmetric algebras (also called Frobenius symmetric algebras to distinguish them from the classical symmetric algebras
$S(V) = T(V)/(\{u \otimes v - v \otimes u \mid u, v \in V\}))$.

# Invariants associated to the first Hochschild cohomology groups

## Symmetric algebras

### Definition

An algebra $A$ is selfinjective if $A$ is isomorphic to its $K$-dual $A^* = \mathrm{Hom}_K(A, K)$ as a left $A$-module.
If $A$ is basic, this is equivalent to the existence of a non-degenerate bilinear form $(,) : A \times A \to K$ that is associative, that is, $(ab, c) = (a, bc)$ for all $a, b, c$ in $A$.

It can be shown that an algebra is selfinjective if, and only if, $A$ is an injective $A$-module or, equivalently, every projective $A$-module is injective.

# Invariants associated to the first Hochschild cohomology groups

**Definition**

An algebra $A$ is symmetric if $A$ is isomorphic to its $K$-dual $A^*$ as an $A$-bimodule.
This is equivalent to the existence of a non-degenerate associative *symmetric* bilinear form $(,) : A \times A \to K$.

A symmetric algebra is selfinjective.

# Invariants associated to the first Hochschild cohomology groups

## Invariants in the centre in positive characteristic

$\mathrm{HH}^0(A) = Z(A)$ is invariant under derived equivalence.

**Külshammer ideals.**

$K$ is a perfect (or algebraically closed) field of characteristic $p > 0$.

**Definition**

$$[A, A] := \mathrm{span}\,\{ab - ba;\, a, b \in A\}\,,$$
$$T_n(A) := \left\{a \in A;\, a^{p^n} \in [A, A]\right\} \quad \text{for } n \in \mathbb{N}.$$

Brauer proved that $(a + b)^{p^n} \equiv a^{p^n} + b^{p^n} \pmod{[A, A]}$ so that $T_n(A)$ is a subspace of $A$.

It is even a $Z(A)$-module and $T_n(A) \subset T_{n+1}(A)$.

# Invariants associated to the first Hochschild cohomology groups

Now assume that $A$ is **symmetric** with symmetric bilinear form $(,)$. Let $M^\perp$ be the orthogonal of a subset $M$ of $A$ for this bilinear form. Then $[A, A]^\perp = Z(A)$.

proof

There is a sequence of ideals in $Z(A)$ :

$$Z(A) = [A, A]^\perp = T_0(A)^\perp \supseteq T_1(A)^\perp \supseteq T_2(A)^\perp \supseteq \cdots \supseteq T_n(A)^\perp \supseteq \cdots$$

called **Külshammer ideals** or **generalised Reynolds ideals**.
They do not depend on the choice of bilinear form on $A$.

They were defined by Külshammer, who also proved that they are Morita invariant.

# Invariants associated to the first Hochschild cohomology groups

$A$ symmetric algebra.

**Theorem (Zimmermann 2007)**

If $B$ is derived equivalent to $A$, then $B$ is necessarily symmetric and the isomorphism between $Z(A)$ and $Z(B)$ induces isomorphisms between $T_n(A)^{\perp}$ and $T_n(B)^{\perp}$ for all $n$.

Bessenrodt, Holm and Zimmermann then defined derived invariants for an algebra $\Lambda$ that is not necessarily symmetric, using the trivial extension algebra of $\Lambda$ which is symmetric.

They are isomorphic to the Külshammer ideals when the algebra $\Lambda$ is symmetric.

# Invariants associated to the first Hochschild cohomology groups

Case of stable equivalences of Morita type.

In general, the centre is not preserved under stable equivalence of Morita type. However,

**Proposition (Liu-Zhou-Zimmermann 2012)**

Let $A$ and $B$ be two algebras that are stably equivalent of Morita type. If $A$ is symmetric then $B$ is symmetric by Liu, and we have $\dim Z(A) = \dim Z(B)$ if, and only if, the number of non-projective simple modules is the same for $A$ and for $B$.

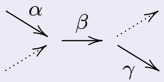
The Auslander-Reiten conjecture states that the number of non-projective simple modules is preserved under stable equivalence.
Pogorzały proved this conjecture for selfinjective *special biserial* algebras.

# Invariants associated to the first Hochschild cohomology groups

## Definition

An algebra $KQ/I$ is special biserial if

- each vertex in $Q$ is the source of at most two arrows and the target of at most two arrows,
- given an arrow $\beta$, there is at most one arrow $\alpha$ whose target is the source of $\beta$ and such that $\alpha\beta \notin I$, and there is at most one arrow $\gamma$ whose source is the target of $\beta$ and such that $\beta\gamma \notin I$.



# Invariants associated to the first Hochschild cohomology groups

In particular,

$$\left.\begin{array}{l} A \text{ special biserial, symmetric, indecomposable} \\ B \text{ special biserial} \\ A,\ B \text{ stably equivalent of Morita type} \end{array}\right\} \implies Z(A) \cong Z(B)$$

(and also $B$ symmetric indecomposable).

# Invariants associated to the first Hochschild cohomology groups

**Stable centre.**

In general, the centre can be replaced by the stable centre.
Recall that $Z(A) \cong \mathrm{End}_{A-A}(A)$.

**Definition**

The stable centre is $Z^{st}(A) = \underline{\mathrm{End}}_{A-A}(A) = \mathrm{End}_{A-A}(A)/Z^{pr}(A)$ where $Z^{pr}(A) = \mathcal{P}(A, A)$ is the projective centre of $A$, formed by the endomorphisms of $A$ that factor through a projective $A$-bimodule.

# Invariants associated to the first Hochschild cohomology groups

**Theorem**

- [Broué] The stable centre is invariant under stable equivalence of Morita type.
- [Liu-Zhou-Zimmermann 2012] If $A$ is symmetric, the ideals $T_n^{st}(A)^{\perp} := T_n(A)^{\perp}/Z^{pr}(A)$ of $Z^{st}(A)$ are invariant under stable equivalence of Morita type.
- [Pan-Zhou 2010] The algebra $\mathrm{HH}^*(A)/Z^{pr}(A)$ is invariant under stable equivalence of Morita type.

# Invariants associated to the first Hochschild cohomology groups

## The Lie structure on the first cohomology group

We have already mentioned that the Lie algebra structure of $\mathrm{HH}^1(A)$ is preserved under derived equivalence.
It is not known in general whether the Lie algebra structure of $\mathrm{HH}^1(A)$ is preserved under stable equivalence of Morita type, except in the following cases:

- [Rouquier] selfinjective algebras in characteristic 0
- [König-Liu-Zhou 2012] symmetric algebras (in this case, the graded Lie algebra $\mathrm{HH}^*(A)/Z^{pr}(A)$ is an invariant).

## Beginning of second talk

In the first talk, we defined derived equivalences and stable equivalences of Morita type.

We described Hochschild cohomology and more specifically

- $\mathrm{HH}^0(A) \cong Z(A)$ and
- $\mathrm{HH}^1(A)$ which is the quotient space of derivations by the inner derivations and is a Lie algebra.

Inside $\mathrm{HH}^0(A)$, we defined the Külshammer ideals when $A$ is a symmetric algebra and $\mathrm{char}(K) > 0$. Külshammer ideals

- They are all invariant under derived equivalence.
- If the algebras are symmetric, the Lie algebra $\mathrm{HH}^1(A)$ is preserved under stable equivalence of Morita type.
- In addition, if the algebras are symmetric, the *stable* Külshammer ideals are invariant under stable equivalence of Morita type.

We shall now use these invariants in order to classify some algebras up to stable equivalence of Morita type.

# Application to the classification of symmetric algebras of dihedral, semi-dihedral and quaternion type

$K$ an algebraically closed field of characteristic $p > 0$.
$G$ a finite group such that $p$ divides $\#G$.
Then the group algebra $KG$ is the direct sum of indecomposable algebras, the blocks of $KG$.
Each block is a symmetric algebra. To each block of $KG$ is associated a *defect group*.
Erdmann studied the representations of blocks of $KG$.
If a block is tame, then its defect group is dihedral, semi-dihedral or generalised quaternion and we must have $p = 2$.

# Application to the classification of symmetric algebras of dihedral, semi-dihedral and quaternion type

Erdmann then introduced and studied finite dimensional symmetric algebras of dihedral, semi-dihedral and quaternion type.
They are characterised by properties of their Auslander-Reiten quiver, their Cartan matrix and their representation type. They contain in particular all tame blocks of group algebras.
She described them up to Morita equivalence, giving a list of representatives by quiver and relations.

# Application to the classification of symmetric algebras of dihedral, semi-dihedral and quaternion type

Derived equivalences.

- [Holm] Classification up to derived equivalence, except some special cases depending on a scalar that he could not separate.
- [Holm-Zimmermann 2008] Use of Külshammer ideals to continue this classification in the case of algebras of dihedral and semi-dihedral type and [Zimmermann 2018] for quaternion type.
  (also [Kauer] for dihedral type, less elementary methods).
  Definition of a specific associative symmetric non-degenerate bilinear form on these algebras (valid for any symmetric algebra defined by quiver and relations).
  It is enough to consider $T_1(A)^\perp$. (Recall that
  $T_1(A) := \{a \in A; a^p \in [A, A]\}$.)
- [Holm-Zhou] Use of Külshammer ideals to prove that a family of algebras in Erdmann's list was indeed in the same derived equivalence class as blocks of group algebras in characteristic 2.

# Application to the classification of symmetric algebras of dihedral, semi-dihedral and quaternion type

Stable equivalences.

Zhou and Zimmermann next studied these algebras up to stable equivalence of Morita type using the invariants in the centre and the stable centre (algebra structure of $Z^{st}(A)/T_n^{st}(A)^{\perp}$).
But these invariants were not enough to separate some families.
I then used the Lie algebra $\mathrm{HH}^1(A)$ to make further progress in separating some of these families.

# Application to the classification of symmetric algebras of dihedral, semi-dihedral and quaternion type

The algebras in Erdmann's classification all have one, two or three simple modules.
$\rightsquigarrow$ 9 subfamilies, each invariant under stable equivalent of Morita type.

For algebras with three simple modules (the only remaining question is for algebras of quaternion type), the study of $\mathrm{HH}^1(A)$ does not give any new information.

# Application to the classification of symmetric algebras of dihedral, semi-dihedral and quaternion type

## One simple module

### Definition

char $K = 2$. Let $k \geqslant 2$ be an integer and set

$$\Lambda := K\langle x, y\rangle/((xy)^k + (yx)^k, (xy)^k x, (yx)^k y).$$

The local algebras of dihedral, semi-dihedral and quaternion type for which there remain questions are:

- dihedral. $\Lambda/(x^2 + (xy)^k, y^2 + d(xy)^k)$ with $d \in \{0, 1\}$.
- semi-dihedral. $\Lambda/(x^2 + (yx)^{k-1}y + c(xy)^k, y^2 + d(xy)^k)$ with $(c, d) \in \{(1, 0); (c, 1),\ c \in K\}$.
- quaternion. $\Lambda/(x^2 + (yx)^{k-1}y + c(xy)^k, y^2 + (xy)^{k-1}x + d(xy)^k)$ with $(c, d) \in K^2$.

# Application to the classification of symmetric algebras of dihedral, semi-dihedral and quaternion type

Already known: if $k \neq k'$, the corresponding algebras are not stably equivalent of Morita type.

## Theorem (T)

The algebras with the following parameters are not stably equivalent of Morita type;

- **Dihedral.** $d = 0$ and $d = 1$.
- **Semi-dihedral.** If $c \neq 0$: $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(c, 1)$.
- **Quaternion.** If $cd \neq 0$: $(0, 0)$, $(0, d)$ and $(c, d)$.
  (NB. $Q(1\mathcal{A})^k(c, d) \cong Q(1\mathcal{A})^k(d, c)$).

Consequence: the classification of the algebras of dihedral type is now complete: the dimension of $\mathrm{HH}^1(A)$ is enough to separate the two families of algebras that remained.

# Application to the classification of symmetric algebras of dihedral, semi-dihedral and quaternion type

Two simple modules

**Quaternion type**: some progress can be made based on a result of Zimmermann's and using stable Külshammer ideals, avoiding the computation of $\mathrm{HH}^1(A)$ (same result!).

# Application to the classification of symmetric algebras of dihedral, semi-dihedral and quaternion type

**Semi-dihedral type**: two simple modules: separation of many (but far from all) algebras using the Lie algebra structure of $\mathrm{HH}^1(A)$.
Ingredients:

- the lower central series,
- the derived series,
- the nilradical,
- the Killing form and
- some generalised derivations (on the Lie algebra $\mathrm{HH}^1(A)$).

The details are very technical and I shall not give them here. They can be found on arxiv:1706.10044 (paper to appear in Homology, Homotopy and Applications).

# Application to the classification of symmetric algebras of dihedral, semi-dihedral and quaternion type

**Remark**

In some cases, the whole algebra $\mathrm{HH}^*(A)$ is known from results of A.I. Generalov, A.A. Ivanov and S.O. Ivanov.
However, I obtain as much information for the purpose distinguishing the algebras up to stable equivalence of Morita type with the Lie structure of $\mathrm{HH}^1(A)$, whose determination is more elementary.

# Application to the classification of generalisations of Nakayama algebras

This is joint work with **Nicole Snashall** [Proc. Edinburgh Math. Soc. 2015].

## Definition

The Nakayama algebras are the algebras such that for any indecomposable projective or injective module $M$, the sequence

$$M \supset \mathrm{Rad}(M) \supset \mathrm{Rad}^2(M) \supset \cdots$$

is a composition series ($M$ is said to be **uniserial**).

# Application to the classification of generalisations of Nakayama algebras

Nakayama algebras have been much studied.

- finite representation type
- module categories well known
- they are the basic algebras such that $\Omega^2_{A-A}(A) \cong A_\sigma$ for some automorphism $\sigma$ [Erdmann-Holm 1999].
- they are the distinct representatives of the stable equivalence classes of Brauer tree algebras [Gabriel-Riedtmann 1979].

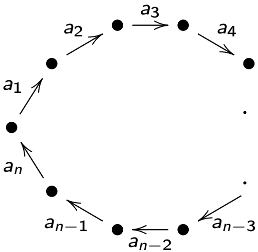
Moreover, the basic Nakayama algebras are special biserial. special biserial

# Application to the classification of generalisations of Nakayama algebras

## Isomorphism classes, quivers and relations

$K$ algebraically closed, $A$ basic $\rightsquigarrow$ $A \cong KQ/I$ with $Q$ a quiver and $I$ a two-sided ideal in $KQ$.
If $e_i$ is a vertex in the quiver, we shall also denote by $e_i$ the path of length 0 at $e_i$.
The basic symmetric Nakayama algebras are the algebras $N_m^n = K\Delta_n/(\text{paths of length} \geqslant nm+1)$.

$\Delta_n$



# Application to the classification of generalisations of Nakayama algebras

They are indeed symmetric; bilinear form

$$(p, 1) = \begin{cases} 1 & \text{if } p \text{ path of length } nm \\ 0 & \text{if } p \text{ path of length } < nm. \end{cases}$$

The paths of length $nm$ are the cyclic permutations of $(a_1 \cdots a_n)^m$, and it is then easy to check that $(,)$ is symmetric. The fact that this bilinear form is non-degenerate and associative is a consequence of a theorem of Holm-Zimmermann in 2008.

# Application to the classification of generalisations of Nakayama algebras

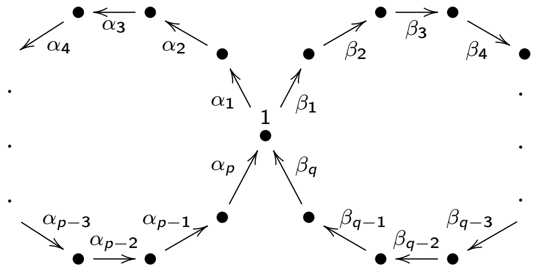
Set $A = KQ/I$ (symmetric) and let $e_1, \ldots, e_n$ be the vertices of $Q$. Then

- the indecomposable projective left $A$-modules are the $Ae_i$;
- $\mathrm{Rad}^k(Ae_i)$ is the vector space generated by the paths of length at least $k$ that start at $e_i$;
- injective modules are projective, so $A$ is a Nakayama algebra if, and only if, every $Ae_i$ is uniserial;
- $Ae_i$ is uniserial if, and only if, for each $k$ there is at most one path of length $k$ that starts at $e_i$.
  This is clearly true for the Nakayama algebras $N_m^n$.

# Application to the classification of generalisations of Nakayama algebras

We were interested in symmetric special biserial algebras with at most one non-uniserial indecomposable projective module. It is easy to see that their quiver must necessarily be $\Delta_n$ or

$Q_{(p,q)},\ 1 \leqslant p \leqslant q$



# Application to the classification of generalisations of Nakayama algebras

Define two ideals in $KQ_{(p,q)}$:

- the ideal $I_r$, for $r \in \mathbb{N}^{>0}$, generated by

$$\begin{aligned}
&\alpha_1\alpha_p, \qquad \beta_1\beta_q, \qquad (\beta_q\cdots\beta_1\alpha_p\cdots\alpha_1)^r - (\alpha_p\cdots\alpha_1\beta_q\cdots\beta_1)^r,\\
&\alpha_i(\alpha_{i-1}\cdots\alpha_1\beta_q\cdots\beta_1\alpha_p\cdots\alpha_i)^r \text{ for all } 2 \leqslant i \leqslant p-1,\\
&\beta_j(\beta_{j-1}\cdots\beta_1\alpha_p\cdots\alpha_1\beta_q\cdots\beta_j)^r \text{ for all } 2 \leqslant j \leqslant q-1;
\end{aligned}$$

- the ideal $J_{(s,t)}$, for $s, t$ in $\mathbb{N}^{>0}$, generated by

$$\begin{aligned}
&\beta_1\alpha_p, \qquad \alpha_1\beta_q, \qquad (\alpha_p\cdots\alpha_1)^s - (\beta_q\cdots\beta_1)^t,\\
&\alpha_i(\alpha_{i-1}\cdots\alpha_1\alpha_p\cdots\alpha_i)^s \text{ for all } 2 \leqslant i \leqslant p-1,\\
&\beta_j(\beta_{j-1}\cdots\beta_1\beta_q\cdots\beta_j)^t \text{ for all } 2 \leqslant j \leqslant q-1,
\end{aligned}$$

  with, if $p = 1$ then $s \geqslant 2$, and, if $q = 1$ then $p = 1, s \geqslant 2$ and $t \geqslant 2$.

# Application to the classification of generalisations of Nakayama algebras

The algebras $KQ_{(p,q)}/I_r$ and $KQ_{(p,q)}/J_{(s,t)}$ are clearly special biserial.

**Remark**

- The algebras $KQ_{(p,q)}/I_1$ and $KQ_{1,n}/J_{(2,2)}$ occur as two of the three families of selfinjective algebras of Euclidean type up to derived and stable equivalence by Bocian-Holm-Skowroński 2004.
- Some of these algebras are derived equivalent to algebras of dihedral type in the classification of Holm:
  - $K\mathcal{Q}_{(1,1)}/I_r = D(1\mathcal{A})_1^r$,
  - $K\mathcal{Q}_{(1,2)}/I_r$ that is derived equivalent to $D(2\mathcal{B})^{1,r}(0)$, and
  - $K\mathcal{Q}_{(2,2)}/I_r$ that is derived equivalent to $D(3\mathcal{K})^{r,1,1}$,

  all three of which come from tame blocks of finite groups when $\mathrm{char}(K) = 2$ and $r$ is a power of 2,

  - and $K\mathcal{Q}_{(2,2)}/J_{(s,t)}$ that is derived equivalent to $D(2\mathcal{R})^{1,s,t,1}$ and which does not come from blocks, see [Holm, Liu].

# Application to the classification of generalisations of Nakayama algebras

**Theorem (Snashall-T)**

The algebras $KQ_{(p,q)}/I_r$ and $KQ_{(p,q)}/J_{(s,t)}$ are symmetric, special biserial, and have at most one non-uniserial indecomposable projective module. Moreover, every basic, indecomposable, finite dimensional, symmetric, special biserial algebra with at most one non-uniserial indecomposable projective module is isomorphic to one of the algebras $N_m^n$, $KQ_{(p,q)}/I_r$ or $KQ_{(p,q)}/J_{(s,t)}$.

# Application to the classification of generalisations of Nakayama algebras

## Remark

- Complete the set $\mathcal{C}$ of cyclic permutations of $(\alpha_p \cdots \alpha_1 \beta_q \cdots \beta_1)^r$ except $(\beta_q \cdots \beta_1 \alpha_p \cdots \alpha_1)^r$ into a basis of $KQ_{(p,q)}/I_r$ consisting of paths; then the non-degenerate symmetric associative bilinear form is defined on these basis elements by $(p, 1) = \begin{cases} 1 & \text{if } p \in \mathcal{C} \\ 0 & \text{if } p \notin \mathcal{C}. \end{cases}$
- Complete the set $\mathcal{C}$ of cyclic permutations of $(\alpha_p \cdots \alpha_1)^s$ and $(\beta_q \cdots \beta_1)^t$ except $(\beta_q \cdots \beta_1)^t$ into a basis of $KQ_{(p,q)}/J_{(s,t)}$ consisting of paths; then the non-degenerate symmetric associative bilinear form is defined on these basis elements by $(p, 1) = \begin{cases} 1 & \text{if } p \in \mathcal{C} \\ 0 & \text{if } p \notin \mathcal{C}. \end{cases}$

Idea of the proof of the theorem

# Application to the classification of generalisations of Nakayama algebras

## Classification up to derived equivalence

**Theorem (Snashall-T)**

- $KQ_{(p,q)}/J_{(s,t)}$ (with $1 \leqslant p \leqslant q$) is derived equivalent to exactly one algebra in the following list:
  - $KQ_{(1,p+q-1)}/J_{(s,t)}$ with $2 \leqslant s \leqslant t$,
  - $N_M^{p+q-1}$ with $p+q>2$ and $\min(s,t)=1$, $\max(s,t)=M$.
- $KQ_{(p,q)}/I_r$ (with $1 \leqslant p \leqslant q$) is derived equivalent to an algebra of the form $KQ_{(p,q)}/J_{(s,t)}$ if and only if they are isomorphic.
  This is only the case for $KQ_{(1,1)}/I_1 \cong KQ_{(1,1)}/J_{(2,2)}$ and char $K \neq 2$.
- $KQ_{(p,q)}/I_r$ and $KQ_{(p',q')}/I_{r'}$ (with $1 \leqslant p \leqslant q$ and $1 \leqslant p' \leqslant q'$) are derived equivalent if and only if $(p,q,r)=(p',q',r')$.

# Application to the classification of generalisations of Nakayama algebras

**Classification up to stable equivalence of Morita type**

**Theorem (Snashall-T)**

- $KQ_{(p,q)}/J_{(s,t)}$ (with $1 \leqslant p \leqslant q$) is stably equivalent of Morita type to exactly one algebra in the following list:
  - $KQ_{(1,p+q-1)}/J_{(s,t)}$ with $2 \leqslant s \leqslant t$,
  - $N_M^{p+q-1}$ with $p+q > 2$ and $\min(s,t) = 1$, $\max(s,t) = M$.
- $KQ_{(p,q)}/I_r$ (with $1 \leqslant p \leqslant q$) is stably equivalent of Morita type to an algebra of the form $KQ_{(p,q)}/J_{(s,t)}$ if and only if they are isomorphic. This is only the case for $KQ_{(1,1)}/I_1 \cong KQ_{(1,1)}/J_{(2,2)}$ and char $K \neq 2$.
- $KQ_{(p,q)}/I_r$ and $KQ_{(p',q')}/I_{r'}$ (with $1 \leqslant p \leqslant q$ and $1 \leqslant p' \leqslant q'$) are stably equivalent of Morita type if and only if $(p,q,r) = (p',q',r')$.

# Application to the classification of generalisations of Nakayama algebras

Proof uses:

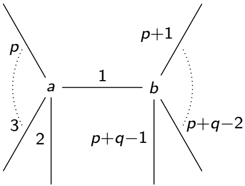
- The number of simple modules ($n$ and $p+q-1$).
- The centre of the algebra (or $\mathrm{HH}^0$), as an algebra. *[Only for derived equivalence.]*
- $\dim \mathrm{HH}^1$.
- The determinant of the Cartan matrix $C_A = (c_{ij})$ where $c_{ij} = e_j A e_i$ [Rickard/Bocian-Skowroński 2007]. *[SEMT: absolute value – [Xi 2008].]*
- $\dim \mathrm{HH}^{2i}$ for $i < p$, computed for $KQ_{(p,q)}/I_r$.
- Külshammer ideals. *[SEMT: $Z^{st}(A)/T_n^{st}(A)^{\perp} \cong Z(A)/T_n(A)^{\perp}$.]*
- Generalised Brauer tree algebras (the Nakayama algebras are derived equivalent to Brauer tree algebras, the others are not). *[SEMT: only for the Nakayama algebras.]*

# Application to the classification of generalisations of Nakayama algebras

Generalised Brauer tree: connected tree, embedded in the plane (so that the edges around a given vertex can be oriented cyclically), whose vertices are endowed with multiplicities, which are positive integers.
To each generalised Brauer tree is associated a basic symmetric algebra, called generalised Brauer tree algebra.
Generalised Brauer tree algebras are completely determined, up to derived equivalence, by the number of edges and the set of multiplicities [Membrillo-Hernández 1997].

# Application to the classification of generalisations of Nakayama algebras

The algebra $KQ_{(p,q)}/J_{(s,t)}$ is the generalised Brauer tree algebra of




with multiplicities $s$ at $a$ and $t$ at $b$.
If all the multiplicities except possibly one of them are equal to 1, the corresponding (generalised) Brauer tree algebras are precisely the algebras which are stably equivalent to a symmetric non-simple Nakayama algebra [Gabriel-Riedtmann 1979]. The multiplicity is $m$.
Membrillo-Hernández' result enabled us to prove the first part of the theorems.

Proof that $[A, A]^\perp = Z(A)$.

$$\begin{aligned}(c, ab - ba) &= (c, ab) - (ba, c)\\ &= (ca, b) - (b, ac)\\ &= (ca, b) - (ac, b)\\ &= (ca - ac, b)\end{aligned}$$

for all $a, b$ in $A$ so

$$c \in [A, A]^\perp \iff ca - ac = 0 \iff c \in Z(A).$$

back

**Idea of the proof of the theorem.**
The basic symmetric algebras for which all indecomposable projectives are uniserial are the Nakayama algebras.
Therefore assume that there is exactly one non-uniserial indecomposable projective, say $Ae_1$.

For a *symmetric* special biserial algebra, the number of arrows into and out of a given vertex are the same (one or two), and given an arrow $\beta$ from $e_i$ to $e_j$, there are *exactly* one arrow $\alpha$ ending at $e_i$ and one arrow $\gamma$ starting at $e_j$ such that $\alpha\beta \notin I$ and $\beta\gamma \notin I$.

All these conditions show that the quiver must be $Q_{(p,q)}$ and gives some conditions on the relations at vertex 1.

For the relations, we use the fact that $A$ is weakly symmetric, and we distinguish two cases:

- $\alpha_1\alpha_p \in I$; we use the fact that $A$ is symmetric to remove scalars and to prove that in the relation that is not a path, the powers of the two cycles must be the same ($r$);
- $\alpha_1\alpha_p \notin I$; here we use the fact that $K$ is algebraically closed to remove scalars.

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